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The Noise from Turbulence Convected at High Speed

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THE NOISE FROM TURBULENCE CONVECTED AT HIGH SPEED

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[Plates 5 and 6]

CONTENTS

The theory initiated by Lighthill (1952) for the purpose of estimating the sound radiated from a turbulent fluid flow is extended to deal with both the transonic and supersonic ranges of eddy convection speed. The sound is that which would be produced by a distribution of convected acoustic quadrupoles whose instantaneous strength per unit volume is given by a turbulence stress tensor, T_{ij}. At low subsonic speeds the radiated intensity increases with the eighth power of velocity although quadrupole convection augments this basic dependence by a factor $|1-M\cos\theta|^{-5}$, where M is the eddy convection Mach number and θ the angular position of an observation point measured from the direction of eddy motion. At supersonic speeds the augmentation factor becomes singular whenever the eddy approaches the observation point at sonic velocity, $M\cos\theta = 1$. At that condition a quadrupole degenerates into its constituent simple sources, for each quadrupole element moves with the acoustic wave front it generates and cancelling contributions from opposing sources, so essential in determining quadrupole behaviour, cannot combine but are heard independently. This simple-source radiation is likened to a type of eddy Mach wave whose strength increases with the cube of a typical flow velocity. When quadrupoles approach the observer with supersonic speed sound is heard in reverse time, but is once again of a quadrupole nature and the general low-speed result is again applicable. The limiting high-speed form of the convection augmentation factor is $|M\cos\theta|^{-5}$ which combines with the basic eighth power velocity law to yield the result that radiation intensity increases only as the cube of velocity at high supersonic speed. The mathematical theory is developed in some detail and supported by more physical arguments, and the paper is concluded by a section where some relevant experimental evidence is discussed.

Introduction

The general theory of sound generated aerodynamically given by Lighthill (1952) has provided a firm basis for the understanding of sound produced by fluctuating airflows in the absence of vibrating solids. By evolving an acoustic analogy Lighthill was able to relate the highly non-linear motion of turbulence to its induced acoustic field with tools already made familiar by the scientists who in the last century created the science of sound. This appeal to classical acoustics brought a comforting sense of familiarity into this entirely new and complicated field and some aspects of sound produced aerodynamically could be inferred by analogy with earlier acoustic studies so well reported by Lord Rayleigh (1896). In this analogy the essentially quadrupole nature of aerodynamic noise is of deep significance, implying a cancellation mechanism brought about by opposing sources of lower order, which makes the generation of sound by aerodynamic means an extremely inefficient process, at least at the lower Mach numbers. These elementary sources are arranged in equal and opposite pairs separated by a distance small compared with the acoustic wavelength. A cancellation effect of this type is discussed by Stokes (1868) when accounting for the diminished efficiency of a bell sounding in a partial vacuum upon the introduction of hydrogen (owing to the augmented acoustic wavelength). The degree of Stokes's effect in that instance was critically dependent on the dipole nature of the bell. Both Rayleigh (1896) and Lamb (1932) in discussing this work emphasized the dependence of the cancellation effect on the source order and their arguments apply with redoubled strength in the understanding of aerodynamic noise.

Although broadly based on well-established principles, the details of Lighthill's theory involved new techniques developed to cope with certain novel aspects of the aerodynamic noise problem. Turbulence, which is a distributed random function of space and time, presents quite a different problem from the harmonically fluctuating point sources of early acoustic theory. The need to relate the quadrupole source strength to details of the turbulence structure imposed a severe demand on the new theory particularly when the distributed quadrupoles and the observer were in relative motion. The acoustic analogy developed in terms of moving reference frames provided a solution to these problems and identified the radiation field as being that due to an equivalent distribution of moving quadrupoles placed in a uniform acoustic medium at rest with the quadrupole source strength density equal to a turbulence stress tensor. In this form Lighthill was able to calculate the acoustic radiation if given a particular specification of the turbulence structure.

Turbulence structures of most practical airflows are little understood, and involved calculations of their sound field are prohibited by our severely restricted knowledge of a complicated turbulence stress tensor. At low speeds, Lighthill (1952) showed that under certain conditions the fluctuating Reynolds stress of a near-incompressible flow can be used as a good approximation to the stress tensor and that the small fluctuations of density can be disregarded without significant error. On this basis Proudman (1952) computed the noise radiated by decaying isotropic turbulence, but his work is probably the only detailed quantitative application of aerodynamic noise theory that is feasible with our present limited understanding of turbulence.

But accurate and involved evaluation of the equations relating sound to its turbulent origin is only one of many uses of the general theory. Perhaps its most important application is in predicting gross effects brought about by only qualitatively-understood features of the complicated turbulent flows found in practice. High-velocity air jets are of particular importance, accounting as they do for most of the noise of modern aircraft. Here the qualitative trends predicted by the theory account for many features observed in their noise fields and allow experiments to be co-ordinated without recourse to empiricism.

The limited understanding of turbulent flows gave developments of theory, that show all features of the noise field deducible from as general as possible a turbulence specification, a particular importance. Lighthill (1954) made use of the relative inefficiency of octupoles to show that a highly directional quadrupole field is produced by shear-flow turbulence. Dimensional reasoning showed the intensity to vary with the eighth power of a typical velocity, and similarity principles were used to demonstrate how jet noise sources are concentrated near the nozzle exit (Ribner 1958). Convection of turbulent eddies generally augments quadrupole efficiency by a factor $(1-M\cos\theta)^{-6}$ (Lighthill 1952), where $M\cos\theta$ is the convection Mach number in the direction of emission based on the atmospheric speed of sound. This factor is modified to $|1-M\cos\theta|^{-5}$ (Ffowcs Williams 1960) to account for a limited source volume, and acoustic frequencies are shifted by a Doppler factor $(1-M\cos\theta)$. These effects are all predicted by making an approximate estimate of the solution to the exact equations for the radiation field, and, since primary concern has been with jets of relatively low velocity, the approximations have been of a type valid at low speeds. These approximate estimates are typified by the most familiar result of Lighthill's theory, the dimensional law for acoustic intensity,

$$I \sim \frac{\bar{\rho}^2 U^8}{\rho_0 a_0^5}\left(\frac{D}{|\mathbf{y}|}\right)^2 \frac{1}{|1-M\cos\theta|^5}.$$

I is written for the intensity, ρ is a typical exhaust gas density, ρ_0 the atmospheric density, a_0 the atmospheric speed of sound, U a typical flow velocity, D a flow dimension and $|\mathbf{y}|$ the distance travelled by the sound wave. This result, now well substantiated by experiment (see Lighthill (1954) and (1961) for discussion), although the U^8 law evidently overestimates somewhat the intensity at higher subsonic speeds, bears witness to the ability of Lighthill's acoustic analogy to yield useful results regarding the noise field of turbulence whose structure is still little understood.

At supersonic flow velocities the problems are even less understood, and, although extrapolation of low-speed theory may provide a crude guide, there is little or no justification for such a step. Indeed the recent work of Phillips (1960) suggests a possible mechanism of sound generation by supersonic shear layers quite different from that indicated by Lighthill (1952) at low speeds. Of course the acoustic analogy is none the less valid, being based on exact equations, but the important question of whether or not it can be usefully exploited at high speeds still remains unanswered.

Objections to its high-speed application are twofold. First, the turbulence stress tensor is a complicated enough parameter at low speeds where the turbulence may effectively be regarded as incompressible; how much more complicated does this become at high speeds in compressible turbulence? The Lighthill retarded-potential solution then becomes an extremely involved and intractable integral equation. This point would be entirely valid were compressible effects on the stress tensor of extreme importance and were turbulence specifications limited to details of velocity fluctuations. In practice, however, it is an unduly pessimistic outlook since density changes in the turbulence only become significant at non-negligible values of the turbulence fluctuation Mach number. In jets where turbulence levels rarely exceed 20 % of the nozzle exit velocity, incompressible turbulence models may be adequate over a relatively wide speed range. Even when turbulence Mach numbers have become significant the presence of density in the source function is not necessarily a disadvantage. Quantitative turbulence studies are still confined to experiments with hot-wire anemometers which sense not a velocity but a Reynolds number, which is a non-dimensional density-velocity product! (see, for example, Kovasznay & Törmack 1950). It may then be that the presence of density in the stress tensor, far from being an embarrassment, can lead to direct application of experimental data where other approaches flounder for lack of a velocity specification. For the present, with our very limited understanding of high Mach number turbulence the density dependence at high speed can only be of academic interest. We are most unlikely to have access to details of either velocity or stress tensor in the near future, and while this state exists the most important developments of theory must be the ones which yield the most general features of acoustic emission, yet make the minimum possible demand on turbulence knowledge. Answers to several important questions should be given by a general theory which regards the turbulence field as specified, and there appears to be no significant loss of generality if Lighthill's stress tensor is once again regarded as the known parameter.

The effect of supersonic eddy convection is not easily predicted and the presence of the singularities associated with $M\cos\theta = 1$ in the low-speed equations is sometimes regarded as a breakdown of the theory, giving rise to the second main objection to a high-speed application of Lighthill's acoustic analogy. That this apparent breakdown exists is confirmation that at high speeds quite a different mechanism of sound generation takes over and the situation may then be analogous, in certain respects, with that described by Phillips (1960) in supersonic shear layers. But even at this apparently singular condition the concept of moving quadrupoles can provide a clear understanding of the new mechanism. At low speeds quadrupole acoustic efficiencies are low due to the near-cancellation brought about by opposing sources of lower order. As the cancellation becomes less complete the radiation increases rapidly but is always subject to the limiting condition that at no time can the quadrupole radiate more efficiently than its constituent elementary sources. This condition would arise if all cancellation ceased. Now when quadrupoles are convected subsonically towards an observer, in order that sound from each element should reach the observer simultaneously, the parts nearer the observer emit at a later time. Since the quadrupoles have moved forward in this time, a distance determined by the convection Mach number, the time delay is further increased and the reduced cancellation, in accordance with the Stokes effect and the increased volume occupied by the source, accounts for the low-speed $|1-M\cos\theta|^{-5}$ factor in the intensity equation. At the condition when $M\cos\theta = 1$, the quadrupole is being convected towards our observer with precisely the speed of sound. The near elements of the quadrupole emit and continue to move with the sound wave they generate. The other quadrupole elements never overtake this wave and are therefore quite unable to make their presence felt. The cancellation effect is then completely absent with our observer hearing sound generated by the elementary sources which constitute the quadrupole. Since simple-source efficiency is higher than that of a quadrupole the acoustic analogy immediately implies that at supersonic convection speeds relatively intense waves should be observed at the eddy Mach angle, $\theta = \cos^{-1} M^{-1}$, an effect now fully substantiated by experiment (Laufer 1961). At even higher convection velocities time delays again take effect and the quadrupole features of aerodynamic noise resume their low-speed significance. At these supersonic speeds frequencies are reversed and cancellation is increased with increasing Mach number, so making the emission less effective. The $|1-M\cos\theta|^{-5}$ factor again becomes valid and its asymptotic form for high Mach number,

M^{-5}, counters the basic $U^8 a_0^{-5}$ factor in Lighthill's dimensional analysis to predict the intensity to vary with U^3 at high supersonic speeds, again an effect consistent with high-speed experimental evidence (Chobotov & Powell 1957).

Of course one can no longer appeal with such success to analogous situations in classical acoustics when applying the acoustic analogy at transonic and supersonic speeds. Early studies on convection were concerned only with the Doppler principle and such novel aspects as the one suggested by Lord Rayleigh where an 'observer would hear a musical piece in correct time and tune, but backwards' were the source to approach the observer with twice the speed of sound. The roar generated by eddies moving supersonically in rocket motor exhausts is certainly heard 'backwards', but is so unmusical that our observer is quite unable to appreciate this wonder envisaged so beautifully by Lord Rayleigh. Multiple sources convected at supersonic speed have been studied by Blokhinstev (1946), and Moretti & Slutsky (1959) have studied convection effects of point sources, but these bear only a limited similarity to the aerodynamic quadrupoles where the Stokes effect plays such an important role. Acoustic emission at the singular condition, $M\cos\theta = 1$, is more analogous to the Mach waves generated by thin aerofoils at supersonic speeds than to any aspect of classical acoustics, yet the acoustic analogy is capable of dealing with even this situation with comparative ease.

The justification of extending Lighthill's theory to high speed must rest on the conceptual simplicity of the approach and not on the benefits to be reaped from analogies with well-understood aspects of earlier studies. This paper deals with the general theory and develops Lighthill's low-speed approximations to present a unified approach valid over a wide range of convection velocity. No account is taken of shock waves which might have important high-speed effects, and the theory would not be expected to deal with hypersonic flow velocities. Comparison with experiment confirms that even in the absence of details of the turbulence stress tensor the theoretical technique developed by Lighthill is capable of predicting many important features in the noise fields of supersonic flows. This experimental confirmation more than justifies any restriction imposed by regarding the turbulence stress tensor as known.

1. The density autocorrelation function

The starting point of the analysis is the formula given by Lighthill (1952) for the noise radiated by a moving eddy structure. From this an expression for the autocorrelation function of the radiated density field is derived for the case of sound generated by convected eddies in the wake of a moving aircraft. The density autocorrelation function is chosen as it provides the basis for calculating both the intensity and spectrum of the radiated field. At zero time separation the function reduces to the mean square density fluctuation while its Fourier transform in time gives the sound power spectral density. The simplification of neglecting small retarded-time differences yields the main Mach number dependence in a form slightly different from the one given by Lighthill (1952). At high convection Mach numbers the neglect of retarded-time differences leads to singular regions but their inclusion leads to the discovery of a régime of flow where the mechanism of sound production is of quite a different form from the now classical low-speed case. This radiation may be interpreted as an emission of eddy Mach waves, and the analysis has similar features to Phillips's (1960) work on supersonic shear layers. The most striking difference between this and the low-speed case is that the Mach-wave emission is quite independent of the temporal development of the turbulence which plays such an important rôle at low speeds.

Lighthill's treatment is based on an acoustic analogy. The turbulence stress tensor, T_{ij}, determines the strength of the acoustic quadrupoles required to produce in a perfect medium at rest the same sound field as is produced in the real atmosphere by the distributed random turbulence with its associated refractive régimes and other flow inhomogeneities. Provided the stress tensor is known, the sound field can be derived very simply.

The far-field moving-axis equations are applicable provided the turbulence is confined to a limited volume within an otherwise infinite homogeneous acoustic medium where no solid surface or matter sources exist. The theory does not deal with the hypersonic régime of flow nor with the effects of shock waves.

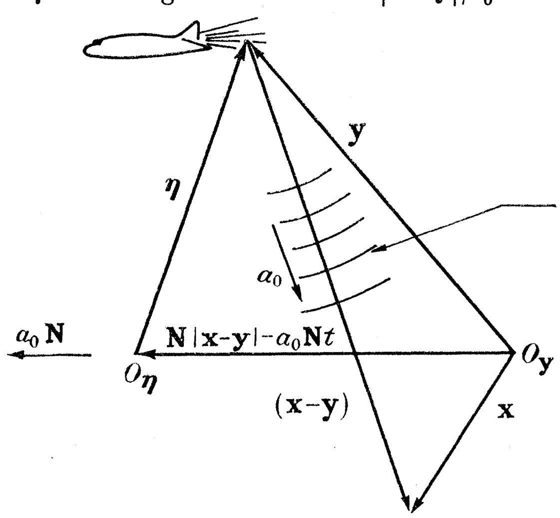

FIGURE 1. Diagram of co-ordinate systems showing the relation between $\mathbf{y}$ and $\boldsymbol{\eta}$ at the instant when sound arriving at the observer at time t was generated. $\mathbf{y} = \boldsymbol{\eta}-a_0\mathbf{N}t+\mathbf{N}|\mathbf{x}-\mathbf{y}|$.

The turbulence will be regarded as specified in a reference frame which moves with the aircraft and this will be the $\boldsymbol{\eta}$ system of co-ordinates. At a constant value of $\boldsymbol{\eta}$ the turbulence is then a statistically stationary function of time.

Consider the aircraft to move with a constant velocity $-a_0\mathbf{N}$, a_0 being the speed of sound in the uniform atmosphere where the aircraft causes the only inhomogeneity. The sound heard at a point $(\mathbf{x}, t)$ was emitted at $\mathbf{y}$ at the retarded time $t-|\mathbf{x}-\mathbf{y}|\,a_0^{-1}$. The origin in $\boldsymbol{\eta}$ moves with the aircraft such that at time $t = 0$ the $\boldsymbol{\eta}$ and $\mathbf{y}$ axes coincide.

$\mathbf{y} = \boldsymbol{\eta}-a_0\mathbf{N}t$ is the relation between the co-ordinate systems when both specify the same point. At the times when the sound reaching our observer was generated the $\mathbf{y}$ and $\boldsymbol{\eta}$ co-ordinates were related by the equation

$$\mathbf{y} = \boldsymbol{\eta}-a_0\mathbf{N}t+\mathbf{N}|\mathbf{x}-\mathbf{y}|. \quad (1\cdot1)$$

This system is illustrated in figure 1.

Lighthill's formula for the density fluctuation, $(\rho-\rho_0)$, can be written

$$\{\rho-\rho_0\}(\mathbf{x},t) \sim \frac{1}{4\pi a_0^4}\int \frac{(x_i-y_i)(x_j-y_j)}{\{|\mathbf{x}-\mathbf{y}|+\mathbf{N}.(\mathbf{x}-\mathbf{y})\}^3}\frac{\partial^2 T_{ij}}{\partial \tau_1^2}(\boldsymbol{\eta},\tau_1)\,\mathrm{d}\boldsymbol{\eta}, \tag{1·2}$$

where the integral must be evaluated over all space at the retarded time:

$$\tau_1 = t-|\mathbf{x}-\mathbf{y}|/a_0.$$

This formula is taken as the basis of the present theory.

The mean density product with time delay τ^*, or the density autocorrelation function, $B(\mathbf{x},t,\tau^*)$ is now formed from the above equation:

$$\overline{\{\rho-\rho_0\}(\mathbf{x},t)\{\rho-\rho_0\}(\mathbf{x},t+\tau^*)} = B(\mathbf{x},t,\tau^*)$$

$$= \frac{1}{16\pi^2 a_0^8}\iint \frac{(x_i-y_i)(x_j-y_j)(x_k-z_k)(x_l-z_l)}{\{|\mathbf{x}-\mathbf{y}|+\mathbf{N}.(\mathbf{x}-\mathbf{y})\}^3\{|\mathbf{x}-\mathbf{z}|+\mathbf{N}.(\mathbf{x}-\mathbf{z})\}^3}$$

$$\times\overline{\frac{\partial^2}{\partial\tau_1^2}T_{ij}(\boldsymbol{\eta},\tau_1).\frac{\partial^2}{\partial\tau_2^2}T_{kl}(\boldsymbol{\xi},\tau_2)}\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\boldsymbol{\xi}. \tag{1·3}$$

Here $\boldsymbol{\xi}$ and $\mathbf{z}$ have a similar significance to $\boldsymbol{\eta}$ and $\mathbf{y}$ and the retarded time

$$\tau_2 = t+\tau^*-|\mathbf{x}-\mathbf{z}|/a_0.$$

Although in general this mean product only exists in the ensemble sense since the passage of the aircraft causes the noise to vary with time, we can nevertheless employ time-averaging techniques. This is made possible by the specification of the turbulence in the $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ space where it is a stationary function of time and by invoking the ergodic hypothesis. We can choose for simplicity to average over a range of τ_1. This averaging process can be used to reduce all the time dependence to a function of the time difference, τ,

$$\tau = \tau_2-\tau_1, \quad \frac{\partial}{\partial\tau_2} = \frac{\partial}{\partial\tau}. \tag{1·4}$$

The required average is then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{\partial^2}{\partial\tau_1^2}T_{ij}(\boldsymbol{\eta},\tau_1).\frac{\partial^2}{\partial\tau^2}T_{kl}(\boldsymbol{\xi},\tau_1+\tau)\,\mathrm{d}\tau_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\frac{\partial^2}{\partial\tau^2}\int_{-T}^{T}\frac{\partial}{\partial\tau_1}\left\{\frac{\partial}{\partial\tau_1}T_{ij}(\boldsymbol{\eta},\tau_1).T_{kl}(\boldsymbol{\xi},\tau_1+\tau)\right\}-\frac{\partial T_{ij}}{\partial\tau_1}(\boldsymbol{\eta},\tau_1).\frac{\partial T_{kl}}{\partial\tau_1}(\boldsymbol{\xi},\tau_1+\tau)\,\mathrm{d}\tau_1. \tag{1·5}$$

The first term in the above average integrates directly and can make no contribution. Now

$$\frac{\partial}{\partial\tau_1}T_{kl}(\boldsymbol{\xi},\tau_1+\tau) = \frac{\partial}{\partial\tau}T_{kl}(\boldsymbol{\xi},\tau_1+\tau), \tag{1·6}$$

a relation that allows us to rewrite the average in the form

$$\lim_{T\to\infty}-\frac{1}{2T}\frac{\partial^3}{\partial\tau^3}\int_{-T}^{T}\frac{\partial}{\partial\tau_1}T_{ij}(\boldsymbol{\eta},\tau_1).T_{kl}(\boldsymbol{\xi},\tau_1+\tau)\,\mathrm{d}\tau_1. \tag{1·7}$$

On repeating the operation this becomes

$$\lim_{T\to\infty}\frac{1}{2T}\frac{\partial^4}{\partial\tau^4}\int_{-T}^{T}T_{ij}(\boldsymbol{\eta},\tau_1)\,T_{kl}(\boldsymbol{\xi},\tau_1+\tau)\,\mathrm{d}\tau_1. \tag{1·8}$$

Since this average is independent of τ_1 we can write

$$\overline{\frac{\partial^2}{\partial \tau_1^2} T_{ij}(\boldsymbol{\eta}, \tau_1) \cdot \frac{\partial^2}{\partial \tau_2^2} T_{kl}(\boldsymbol{\xi}, \tau_2)} = \frac{\partial^4}{\partial \tau^4} R^*_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\xi}, \tau), \tag{1·9}$$

where $R^*_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\xi}, \tau)$ is the stress-tensor correlation function.

$$R^*_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\xi}, \tau) = \overline{T_{ij}(\boldsymbol{\eta}, \tau_1)\, T_{kl}(\boldsymbol{\xi}, \tau_1+\tau)}.$$

For convenience we now introduce the separation vector $\boldsymbol{\Delta}$:

$$\boldsymbol{\xi} = \boldsymbol{\eta} + \boldsymbol{\Delta}, \quad \mathrm{d}\boldsymbol{\xi} = \mathrm{d}\boldsymbol{\Delta}.$$

By defining the correlation function

$$R_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\Delta}, \tau) = R^*_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\xi}, \tau), \tag{1·10}$$

the density autocorrelation function, $B(\mathbf{x}, t, \tau^*)$, can be written in terms of the independent variables $\mathbf{x}, t, \boldsymbol{\eta}$ and $\boldsymbol{\Delta}$:

$$B(\mathbf{x}, t, \tau^*) \sim \frac{1}{16\pi^2 a_0^8} \iint \frac{(x_i - y_i)(x_j - y_j)(x_k - z_k)(x_l - z_l)}{\{|\mathbf{x}-\mathbf{y}| + \mathbf{N}.(\mathbf{x}-\mathbf{y})\}^3 \{|\mathbf{x}-\mathbf{z}| + \mathbf{N}.(\mathbf{x}-\mathbf{z})\}^3} \frac{\partial^4}{\partial \tau^4} R_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\Delta}, \tau)\, \mathrm{d}\boldsymbol{\eta}\, \mathrm{d}\boldsymbol{\Delta}, \tag{1·11}$$

where the integrals are to be evaluated over all space with repeated suffices summed over 1, 2 and 3 at the correct retarded-time difference given by

$$\tau = \tau_2 - \tau_1 = \tau^* + \frac{|\mathbf{x}-\mathbf{y}| - |\mathbf{x}-\mathbf{z}|}{a_0}. \tag{1·12}$$

In jet flows turbulent eddies grow as they move away from the nozzle exit and decay as they leave the mixing region. Their net motion is usually in the opposite direction to the aircraft but may vary from point to point. As eddies only exist over a small region in space non-zero values of the correlation function are confined to small separations of $\mathbf{y}$ and $\mathbf{z}$. We can then neglect the small variations of $(\mathbf{x}-\mathbf{y})$ and $(\mathbf{x}-\mathbf{z})$ over the correlation range and limit the approach to large values of $(\mathbf{x}-\mathbf{y})$, or the acoustic far field. In this far field the difference $\{|\mathbf{x}-\mathbf{y}| - |\mathbf{x}-\mathbf{z}|\}$ can be written, $\boldsymbol{\delta}.(\mathbf{x}-\mathbf{y})/|\mathbf{x}-\mathbf{y}|$, where $\boldsymbol{\delta}$ is the separation vector $(\mathbf{z}-\mathbf{y})$. The correct retarded time can then be regarded as a function of $\boldsymbol{\Delta}$, the separation variable in the correlation tensor:

$$\boldsymbol{\delta} = \boldsymbol{\Delta} - a_0 \mathbf{N}\tau = \mathbf{z} - \mathbf{y},$$

$$\tau = \tau^* + \frac{|\mathbf{x}-\mathbf{y}| - |\mathbf{x}-\mathbf{z}|}{a_0} = \tau^* + \frac{\boldsymbol{\delta}.(\mathbf{x}-\mathbf{y})}{a_0\,|\mathbf{x}-\mathbf{y}|} = \frac{a_0\tau^*|\mathbf{x}-\mathbf{y}| + \boldsymbol{\Delta}.(\mathbf{x}-\mathbf{y})}{a_0(|\mathbf{x}-\mathbf{y}| + \mathbf{N}.(\mathbf{x}-\mathbf{y}))}. \tag{1·13}$$

Although the important effects of convection can be isolated mathematically by a further transformation of variables, and this will be done at a later stage, some aspects of the problem are more simply understood if these effects are first demonstrated by more physical arguments. These arguments centre on an approximate estimate of an exact equation for the mean square density, $B(\mathbf{x}, t, 0)$, but it is essential in attempting the estimate that the quadrupole nature of the phenomenon and the relevant Stokes effect be taken into proper account. The effect of eddy convection on the acoustic intensity will be developed here, very briefly, by considering some general properties of a typical turbulence correlation function. This technique, outlined by Lighthill (1961), is included here to demonstrate qualitatively

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the major effects of convection and to serve as an introduction to the more general and rigorous approach that follows. It will be found convenient to rewrite equation (1·11) in the modified form

$$B(\mathbf{x},t,0) \sim \frac{1}{16\pi^2 a_0^8}\iint \frac{1}{|\mathbf{x}-\mathbf{y}|^2\{1+N\cos\phi\}^6}\frac{\partial^4}{\partial\tau^4}\Lambda(\boldsymbol{\eta},\Delta_x,\tau)\,\mathrm{d}\Delta_x\,\mathrm{d}\boldsymbol{\eta}. \quad (1{\cdot}14)$$

This equation is to be evaluated with $\tau = \Delta_x a_0^{-1}(1+N\cos\phi)^{-1}$, where $|\mathbf{x}-\mathbf{y}|\,N\cos\phi$ has been written for the scalar product $\mathbf{N}.(\mathbf{x}-\mathbf{y})$. This suffix, x, implies the component in the direction of emission and $\Lambda(\boldsymbol{\eta},\Delta_x,\tau)$ represents the integral of the correlation tensor, $R_{xxxx}(\boldsymbol{\eta},\boldsymbol{\Delta},\tau)$ (no summation is implied here), over the surface $\boldsymbol{\Delta}.(\mathbf{x}-\mathbf{y}) = \Delta_x|\mathbf{x}-\mathbf{y}|$. Since this is an instantaneous integral it cannot be affected by eddy convection and is therefore free of convection Mach number effects.

Now $\Lambda(\boldsymbol{\eta},\Delta_x,\tau)$ is a typical space time correlation function with a maximum along some convection velocity line, $\Delta_x = U_c\tau$. U_c is the eddy convection velocity which in this case is best written as $a_0\{N\cos\phi+M\cos\theta\}$ since most generally eddies move downstream relative to the observer with a velocity $a_0 M$, say, and the velocity in the oblique direction of emission is its resolved component, $a_0 M\cos\theta$. Relative to the aircraft, where this correlation is defined, the convection velocity is increased by the component of aircraft velocity in the direction of emission, $a_0 N\cos\phi$, to the value $a_0\{N\cos\phi+M\cos\theta\}$. Figure 2 shows three typical constant correlation contours of the function $\Lambda(\boldsymbol{\eta},\Delta_x,\tau)$ which might be found in convected turbulence. These three diagrams correspond to:

(1) subsonic eddy convection towards observer ($M\cos\theta \ll 1$);
(2) sonic eddy convection towards observer ($M\cos\theta = 1$);
(3) supersonic eddy convection towards observer ($M\cos\theta \gg 1$).

The first and third cases are very similar in that integration along the retarded time line, $\tau = \Delta_x a_0^{-1}(1+N\cos\phi)^{-1}$, effectively increases the space scale by a factor

$$|1+N\cos\phi|\,|1-M\cos\theta|^{-1}$$

and the time range over which the eddy emits by a factor $|1-M\cos\theta|^{-1}$. These increased ranges reduce the cancellation and so enhance the radiation according to the Stokes effect, but it is extremely difficult to deduce the degree of enhancement by simple observation. This difficulty is entirely due to the essentially complicated structure of the quadrupole source and our inability to visualize effectively the mechanism of reduced cancellation. That the now familiar Lighthill factor is present is readily demonstrated by an easy though somewhat tedious computation of the integral for a correlation function of a type consistent with quadrupole sources. To be consistent the correlation must vanish together with its first three derivatives at large separations in either space or time, have a non-vanishing instantaneous integral, if higher-order sources are not to become significant (Lighthill 1952), and must demonstrate convective features. A suitable form would be

$$\frac{\partial^4}{\partial\tau^4}\Lambda(\boldsymbol{\eta},\Delta_x,\tau) = \frac{\partial^4}{\partial\tau^4}\Lambda(\boldsymbol{\eta},0,0)\exp-\{l^{-2}(\Delta_x-[N\cos\phi+M\cos\theta]\,a_0\tau)^2+\omega^2\tau^2\}, \quad (1{\cdot}15)$$

where l and ω are a typical turbulence length and frequency respectively. This function integrates over Δ_x at the retarded time, $\Delta_x a_0^{-1}(1+N\cos\phi)^{-1}$, to give

$$\Lambda(\boldsymbol{\eta},0,0)\,|1+N\cos\phi|^5\,12\sqrt{\pi}\,l\omega^4\{(1-M\cos\theta)^2+l^2\omega^2/a_0^2\}^{-\frac{5}{2}}. \quad (1{\cdot}16)$$

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When $(1-M\cos\theta)^2 \gg l^2\omega^2 a_0^{-2}$, a condition usually found in practice (see § 5), the convective effects introduce a factor $|1+N\cos\phi|^{-1}|1-M\cos\theta|^{-5}$ into the equation for $B(\mathbf{x}, t, 0)$ and we shall show at a later stage that this is a perfectly general result.

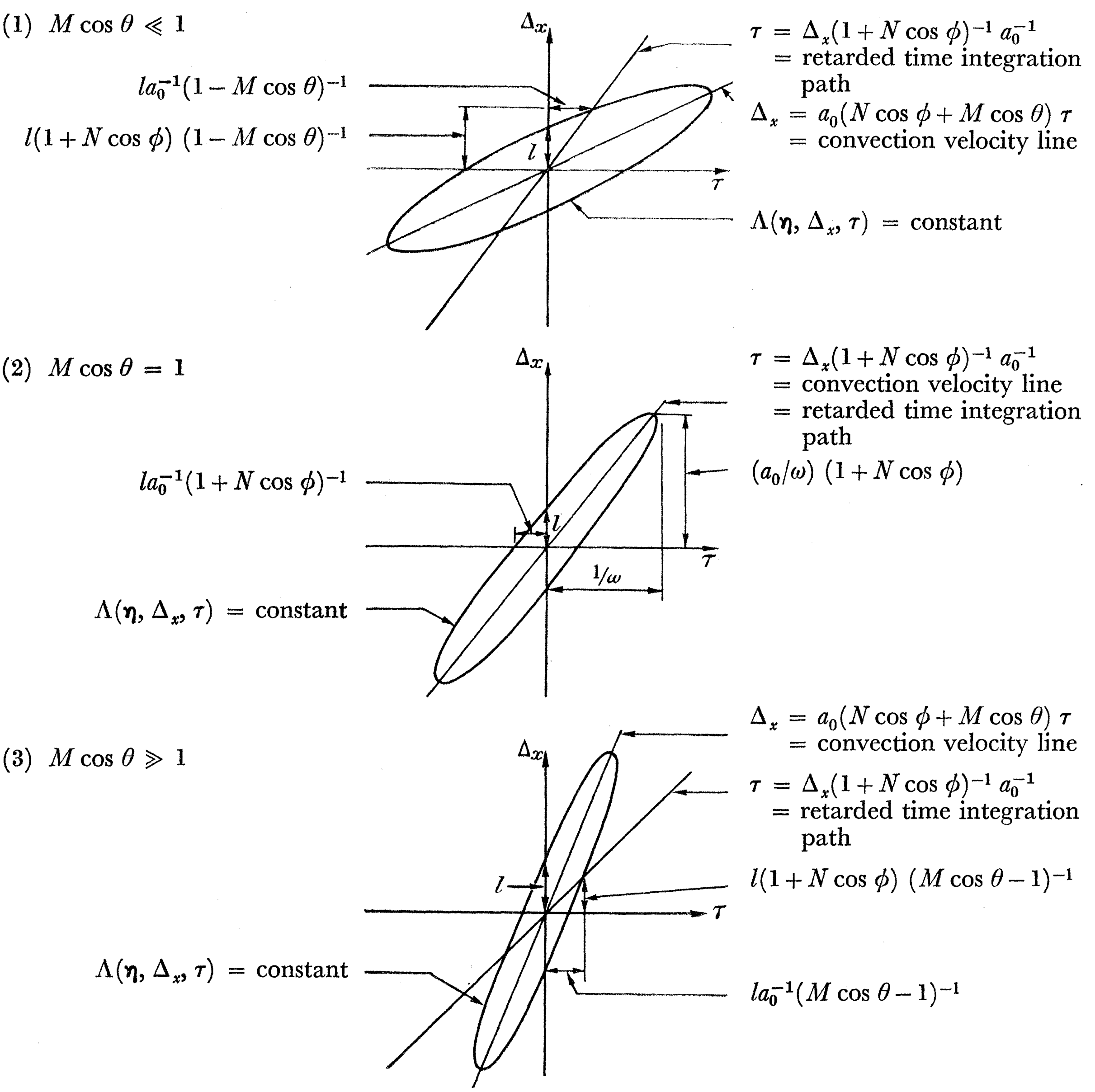

FIGURE 2. Three typical constant correlation curves of the function $\Lambda(\eta, \Delta_x, \tau)$ showing convection velocity lines, integration paths at retarded time and the more important scale changes brought about by convection.

Approximate low-speed theory predicts a singular result when $M\cos\theta = 1$, but this particular correlation function shows the peak value to be

$$|1+N\cos\phi|^5\, 12\sqrt{\pi}(a_0^5/l^4\omega)\,\Lambda(\eta, 0, 0). \tag{1·17}$$

But it is quite unnecessary to rely on examples for the finite value since this condition is the one instance when approximate methods can give accurate results. This is due to the fact that the eddy is approaching our observer at precisely the speed of sound which, as we have already seen, is no longer a case of classical quadrupole emission where the Stokes effect accounts for important trends which may be far from obvious. The observer now hears the sound of the elementary simple sources which constitute the quadrupole, and we are no longer in any difficulty when trying to estimate the magnitude of the radiation. This situation is illustrated in the second diagram of figure 2. The time scale is evidently $la_0^{-1}(1+N\cos\phi)^{-1}$ and the integration scale is $a_0\,\omega^{-1}\,|1+N\cos\phi|$. Together these allow the correlation integral to be estimated very simply as

$$\int\frac{\partial^4}{\partial\tau^4}\Lambda(\boldsymbol{\eta},\Delta_x,\tau)\,\mathrm{d}\Delta_x\sim\Lambda(\boldsymbol{\eta},0,0)\,\frac{a_0^4(1+N\cos\phi)^4}{l^4}\,\frac{a_0}{\omega}\,|1+N\cos\phi|$$

$$\sim|1+N\cos\phi|^5\,\frac{a_0^5}{l^4\omega}\,\Lambda(\boldsymbol{\eta},0,0). \tag{1·18}$$

Typically $\Lambda(\boldsymbol{\eta},0,0)$ is $\overline{T^2}V/l$, where $\overline{T^2}$ is the mean square fluctuation of the stress tensor, T_{ij}, and V is the correlation volume. The acoustic power generated by unit volume of turbulence which is proportional, symbolically, to

$$\frac{\partial}{\partial\boldsymbol{\eta}}B(\mathbf{x},t,0)\,|\mathbf{x}-\mathbf{y}|^2\,\frac{a_0^3}{\rho_0}, \tag{1·19}$$

is readily derived.

At the condition $M\cos\theta=1$ this is very simply

$$\frac{1}{|1+N\cos\phi|}\,\frac{\overline{T^2}V}{\rho_0\,\omega l^5}, \tag{1·20}$$

which differs slightly from the value given by Lighthill (1961) who did not consider the influence of convection on the effective length in the cross-stream direction.

More generally, at both subsonic and supersonic speeds, when $(1-M\cos\theta)^2\gg l^2\omega^2a_0^{-2}$, the energy generated per unit volume is the more familiar quantity

$$\frac{1}{|1+N\cos\phi|}\,\frac{1}{|1-M\cos\theta|^5}\,\frac{\overline{T^2}\omega^4V}{\rho_0\,a_0^5}. \tag{1·21}$$

These are two important results which can be derived directly from equation (1·11), but only in the case $M\cos\theta=1$ is the procedure straightforward. The main difficulty stems from the various subtleties associated with the quadrupole source function and these difficulties are not entirely conceptual. Although equation (1·11) is perfectly valid, it suffers the disadvantage of being very sensitive to small errors. Any approximation at that stage, either theoretical or arising from the substitution of experimental data, could lead to very misleading conclusions. The reason for this extreme sensitivity is that in a convected eddy structure the time derivative of the correlation tensor can be written approximately as the convection velocity times the space derivative. Such space derivatives, which may be very large, make no contribution to the integral. The computation of the radiated sound field on the basis of equation (1·11) would then be extremely difficult as only the very small fraction of the integrand which does not integrate to zero produces sound. Great care must be taken that this small part is not discarded in approximations that appear valid when compared to the size of the stress tensor but may be quite unacceptable when compared to the smaller noise-producing elements.

In fact the equation as formed in (1·11) is not at all suited to convected fields in view of the large part of the integrand which makes no contribution to the integral. This difficulty can be overcome by performing another variable change which can also be used to illustrate the effect of eddy convection. This transformation involves the definition of a separation vector $\boldsymbol{\lambda}$ whose origin moves with the turbulent eddy

$$\boldsymbol{\Delta} = \boldsymbol{\lambda} + a_0(\mathbf{M}+\mathbf{N})\,\tau, \tag{1·22}$$

where $\mathbf{M}$ is the eddy convection Mach-number vector relative to $\mathbf{x}$. We now define a moving axis correlation tensor $P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)$ and specify retarded time in terms of $\boldsymbol{\lambda}$:

$$P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau) = R_{ijkl}(\boldsymbol{\eta},\boldsymbol{\Delta},\tau), \tag{1·23}$$

$$\tau = \frac{\boldsymbol{\lambda}\,.\,(\mathbf{x}-\mathbf{y})+a_0\,\tau^*|\mathbf{x}-\mathbf{y}|}{a_0\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}\,.\,(\mathbf{x}-\mathbf{y})\}},\quad \frac{\partial\tau}{\partial\lambda_n} = \frac{(x_n-y_n)}{a_0\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}\,.\,(\mathbf{x}-\mathbf{y})\}}. \tag{1·24}$$

The time derivatives of $P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)$ and $R_{ijkl}(\boldsymbol{\eta},\boldsymbol{\Delta},\tau)$ are not very simply related in view of the dependence of τ on other variables. For completeness these relations are developed here:

$$\begin{aligned}\frac{\partial}{\partial\tau}R_{ijkl}(\boldsymbol{\eta},\boldsymbol{\Delta},\tau) &= \left\{\frac{\partial}{\partial\tau}+\frac{\partial\lambda_n}{\partial\tau_{\Delta=\text{const.}}}\frac{\partial}{\partial\lambda_n}\right\}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)\\ &= \left\{\frac{\partial}{\partial\tau}-a_0(M_n+N_n)\frac{\partial}{\partial\lambda_n}\right\}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau).\end{aligned} \tag{1·25}$$

The suffix, n, signifies the component in the n direction and repeated suffices are to be summed over 1, 2 and 3.

As τ must be regarded as a function of $\boldsymbol{\lambda}$ when evaluating the integral of (1·11), care must be exercised in the partial differentiations required in (1·25):

$$\begin{aligned}\frac{\partial}{\partial\lambda_n}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau=\phi\boldsymbol{\lambda}) &= \left\{\frac{\partial}{\partial\lambda_n}+\frac{\partial\tau}{\partial\lambda_n}\frac{\partial}{\partial\tau}\right\}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)\\ &= \left\{\frac{\partial}{\partial\lambda_n}+\frac{(x_n-y_n)}{a_0\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}\,.\,(\mathbf{x}-\mathbf{y})\}}\frac{\partial}{\partial\tau}\right\}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau).\end{aligned} \tag{1·26}$$

The relation between the time derivatives follows from (1·25) and (1·26):

$$\frac{\partial}{\partial\tau}R_{ijkl}(\boldsymbol{\eta},\boldsymbol{\Delta},\tau) = \left\{\left[\frac{|\mathbf{x}-\mathbf{y}|+\mathbf{N}\,.\,(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|-\mathbf{M}\,.\,(\mathbf{x}-\mathbf{y})}\right]\frac{\partial}{\partial\tau}-a_0(M_n+N_n)\frac{\partial}{\partial\lambda_n}\right\}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau=\phi\boldsymbol{\lambda}). \tag{1·27}$$

The second term on the right-hand side of this equation, when inserted into the integrand of (1·11) can make no contribution to the integral over correlation volume. This is easily shown by completing the divergence to form a surface integral at large values of separation where the correlation function must be zero.

Positive volume elements are related by the Jacobian

$$\mathrm{d}\boldsymbol{\Delta} = \mathrm{d}\boldsymbol{\lambda}\left|\frac{|\mathbf{x}-\mathbf{y}|+\mathbf{N}\,.\,(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|-\mathbf{M}\,.\,(\mathbf{x}-\mathbf{y})}\right|. \tag{1·28}$$

By making use of these relations the density autocorrelation function can be written in a much more useful form:

$$B(\mathbf{x},t,\tau^*) \sim \frac{1}{16\pi^2 a_0^8}\iint\frac{(x_i-y_i)\,(x_j-y_j)\,(x_k-y_k)\,(x_l-y_l)}{\big||\mathbf{x}-\mathbf{y}|+\mathbf{N}\,.\,(\mathbf{x}-\mathbf{y})\big|\,\big||\mathbf{x}-\mathbf{y}|-\mathbf{M}\,.\,(\mathbf{x}-\mathbf{y})\big|^5}\frac{\partial^4}{\partial\tau^4}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)\,\mathrm{d}\boldsymbol{\lambda}\,\mathrm{d}\boldsymbol{\eta}. \tag{1·29}$$

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This is to be evaluated with $$\tau = \frac{a_0\tau^*|\mathbf{x}-\mathbf{y}|+\boldsymbol{\lambda}.(\mathbf{x}-\mathbf{y})}{a_0\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}}.$$

This equation for the acoustic radiation from convected turbulence in the wake of an aircraft in flight will be used to examine in more detail the acoustic efficiency at both high and low speeds. It is more revealing than the preceding equation, (1·11), on two main points.

First, contributions to the integrand from convected patterns, which generate no sound at subsonic speeds, have been minimized, allowing a realistic estimate of the acoustic strength to be obtained from the magnitude of the covariance $P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)$. Errors involved in approximations at this stage have also been substantially reduced by emphasizing that small part of the turbulence which generates sound.

Secondly, and this is perhaps the more significant point, the main dependence of the radiation on convection is shown quite clearly in the $||\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})|^{-5}$ factor. The time scale of the correlation tensor $P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)$ has been maximized by the co-ordinate transformation and small variations in τ can have very little effect on the value of the correlation tensor. This tensor is then fairly insensitive to convection velocity changes and the main effect of convection is immediately apparent. Of course this is not true for large variations in retarded time such as occur near the point $|\mathbf{x}-\mathbf{y}| = \mathbf{M}.(\mathbf{x}-\mathbf{y})$, and it is a fact that retarded-time differences in the moving co-ordinate system are generally larger than those in a fixed frame for equal space separations, but this does not alter the important result that for small separations retarded time can be disregarded. This step has proved invaluable in understanding the strong preference for downstream emission and the complicated influence of retarded time in equation (1·11) is very simply accounted for by the Mach number factors of equation (1·29).

Now $(\mathbf{x}-\mathbf{y})$ is a function of $\boldsymbol{\eta}$ defined by equation (1·1). This can be transformed to give $(\mathbf{x}-\mathbf{y})$ explicitly in terms of the independent variables and the equations become very reminiscent of the well-known solution to the convected wave equation:

$$(\mathbf{x}-\mathbf{y}) = (\mathbf{x}-\boldsymbol{\eta}+a_0\mathbf{N}t)+\frac{\mathbf{N}}{(1-|\mathbf{N}|^2)}$$
$$\times[\mathbf{N}.(\mathbf{x}-\boldsymbol{\eta}+a_0\mathbf{N}t)\mp\surd\{[\mathbf{N}.(\mathbf{x}-\boldsymbol{\eta}+a_0\mathbf{N}t)]^2+(1-|\mathbf{N}|^2)\,|\mathbf{x}-\boldsymbol{\eta}+a_0\mathbf{N}t|^2\}], \quad (1{\cdot}30)$$
$$|\mathbf{x}-\mathbf{y}|+\mathbf{N}.(\mathbf{x}-\mathbf{y}) = \pm\surd\{[\mathbf{N}.(\mathbf{x}-\boldsymbol{\eta}+a_0\mathbf{N}t)]^2+(1-|\mathbf{N}|^2)\,|\mathbf{x}-\boldsymbol{\eta}+a_0\mathbf{N}t|^2\}. \quad (1{\cdot}31)$$

The significance of the two roots in these equations is that sound reaching an observer at any one instant was emitted by the aircraft at two different positions. The coefficient of $\mathbf{N}$ in the second term on the right-hand side of equation (1·30) is equal to $-|\mathbf{x}-\mathbf{y}|$ which must always be negative. Only one root exists which satisfies this condition at subsonic aircraft speeds, $|\mathbf{N}| < 1$, but at higher speeds sound reaching an observer inside the Mach cone at one instant comprises two rays of sound emitted from the same position relative to the aircraft but at different times and different points, $\mathbf{y}_1$ and $\mathbf{y}_2$, say. The positions at which the aircraft emits sound to reach the observer at the instant considered are easily calculated with the formula (1·30).

The integration over $\boldsymbol{\eta}$ involves a change in the value of $\mathbf{y}$, but as $\boldsymbol{\eta}$ is restricted to a small range by the limited source volume, so is $\mathbf{y}$ in most cases and $(\mathbf{x}-\mathbf{y})$ can be treated as constant. This however is not permissible at all times since even small changes in $\boldsymbol{\eta}$

sometimes require very large changes in $\mathbf{y}$ and under those conditions the variation of $(\mathbf{x}-\mathbf{y})$ throughout the volume of integration would have to be considered. If $\boldsymbol{\eta}$ is to vary over a small range L, then $\mathbf{y}$ will have to vary over a range $L\,\partial\mathbf{y}/\partial\boldsymbol{\eta}$, and the fractional change in $\mathbf{y}$ can be written as

$$\frac{\Delta\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \sim \frac{L}{|\mathbf{x}-\mathbf{y}|}\frac{\partial\mathbf{y}}{\partial\boldsymbol{\eta}} \sim \frac{L}{|\mathbf{x}-\mathbf{y}|}\frac{|\mathbf{x}-\mathbf{y}|}{\{|\mathbf{x}-\mathbf{y}|+\mathbf{N}.(\mathbf{x}-\mathbf{y})\}}. \tag{1·32}$$

The value of $(\mathbf{x}-\mathbf{y})$ remains relatively unchanged provided the ratio

$$L\{|\mathbf{x}-\mathbf{y}|+\mathbf{N}.(\mathbf{x}-\mathbf{y})\}^{-1}$$

remains small. This is usually the case in the far field at large $|\mathbf{x}-\mathbf{y}|$. But when $|\mathbf{N}| \geqslant 1$ the length $\{|\mathbf{x}-\mathbf{y}|+\mathbf{N}.(\mathbf{x}-\mathbf{y})\}$ can become zero, implying an infinite change in $\mathbf{y}$ for a fractional change in $\boldsymbol{\eta}$. Under these conditions $(\mathbf{x}-\mathbf{y})$ must be treated as a variable for $\boldsymbol{\eta}$ integration.

In most practical applications of the theory (excluding the surface of the aircraft Mach cone) changes in $(\mathbf{x}-\mathbf{y})$ can be discounted and the density autocorrelation function is given by equation (1·29) evaluated at constant $(\mathbf{x}-\mathbf{y})$.

Although the choice of $\mathbf{M}$ is based on the velocity of the reference frame in which the turbulence has its maximum time scale, the analysis is perfectly valid for any value of $\mathbf{M}$. This point is important as it implies that if a reference frame exists in which the turbulence does not change, so that time derivatives vanish, the integral is identically zero. This is a proof of the well-known fact that a pattern of turbulence which is convected at a uniform velocity without distortion of its spatial properties will radiate no sound.

From a casual glance at equation (1·29) it appears that this is true over the entire speed range—but this is not so. At supersonic convection velocities the integrand can become singular as $\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}$ approaches zero. Here the zero of the correlation function time derivative associated with an unchanging pattern divided by the zero causing the singularity can have a finite value. Indeed it has, and the radiation emitted takes the form of eddy Mach waves which are analogous to the Mach waves generated by thin supersonic aerofoils. This is the mathematical analogue to the fundamental change of mechanism brought about when the quadrupole source approaches the observer with exactly sonic velocity. The Mach waves are the simple-source-like emission which has been predicted by the preceding physical arguments.

Turbulence moving at supersonic speeds generates an array of Mach waves whose strength is little dependent on the temporal development of turbulence as it moves downstream. This is in vivid contrast to the quadrupole emission where the temporal development is probably the most important single factor determining acoustic efficiency.

A more detailed study of the Mach wave emission is discussed below and this corresponds in part to the situation treated by Phillips (1960).

2. The singularities of the autocorrelation equation

The equation governing the density autocorrelation as written in (1·29) requires specialized treatment whenever apparent singularities arise. One such situation occurs when the aircraft is flying at supersonic speeds and the relative position of the observer to the aircraft is such that he lies on the aircraft Mach cone:

$$||\mathbf{x}-\mathbf{y}|+\mathbf{N}.(\mathbf{x}-\mathbf{y})| = 0. \tag{2·1}$$

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Here the integrand of equation (1·29) becomes infinite and the dependence of $(\mathbf{x}-\mathbf{y})$ on $\boldsymbol{\eta}$ has to be considered. It is easily shown that the integral over $\boldsymbol{\eta}$ remains finite by expanding the integrand about the singularity, but it is doubtful whether this condition can ever have any practical significance. At this singular point the observer hearing the sound is on the edge of the Mach cone which is close to the shock cone generated by the aircraft's motion. The pressure difference caused by the passage of this shock wave must be so large as to make the Mach wave due to the noise completely insignificant. In view of this we shall confine discussion of this particular singularity to noting that the integral remains finite and that it is not a case of great practical importance. We can then limit our attention to regions away from this singularity where, as was pointed out in the previous section, changes in $(\mathbf{x}-\mathbf{y})$ can safely be ignored over the integration range of $\boldsymbol{\eta}$ at both supersonic and subsonic speeds. At supersonic speeds, however, each point in the jet contributes to the sound heard at any instant twice, and the variation of $(\mathbf{x}-\mathbf{y})$ between the two times of emission must be considered. This is very easily done by choosing the two roots of equation (1·30).

The second and most important singularity occurs at supersonic eddy convection speeds when the observer is on the Mach cone generated by the moving eddies,

$$||\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})| = 0. \tag{2·2}$$

This is a case of considerably more interest and can occur at all aircraft speeds. It will be shown that the integrand remains finite at this condition because the correlation function is zero. Physically this is of interest as it corresponds to an emission of what have been termed 'eddy Mach waves'. In this approach to the theory of sound production the equations only indicate this phenomenon specifically at the singularity, although, since the frame of reference could be chosen arbitrarily, Mach wave emission could be demonstrated over a range of supersonic convection speeds. Such a procedure would be more analogous to the analysis developed by Phillips (1960). In this approach the theory deals with both the quadrupole and Mach wave emission as being essentially the same phenomenon, the Mach wave case being merely a simple-source emission by quadrupoles approaching the observer at the speed of sound.

The behaviour of the autocorrelation equation in this region can be examined by considering the integral of the correlation tensor

$$\int \frac{\partial^4}{\partial \tau^4} P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)\,\mathrm{d}\boldsymbol{\lambda}, \tag{2·3}$$

it being remembered that the retarded time-difference, τ, must play an important part.

Defining the four-dimensional Fourier transform of the correlation tensor as the power spectral density tensor

$$P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\sigma) = \iint H_{ijkl}(\boldsymbol{\eta},\mathbf{k},\omega)\,\mathrm{e}^{\mathrm{i}\omega\sigma}\,\mathrm{e}^{\mathrm{i}\mathbf{k}.\boldsymbol{\lambda}}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\omega, \tag{2·4}$$

and noting that differentiation is equivalent to a frequency weighting

$$(\mathrm{i}\omega)^n H_{ijkl}(\boldsymbol{\eta},\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\iint \frac{\partial^n}{\partial\sigma^n} P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\sigma)\,\mathrm{e}^{-\mathrm{i}\omega\sigma}\,\mathrm{e}^{-\mathrm{i}\mathbf{k}.\boldsymbol{\lambda}}\,\mathrm{d}\boldsymbol{\lambda}\,\mathrm{d}\sigma, \tag{2·5}$$

while the retarded-time difference is a dependent variable

$$\tau = \frac{a_0\,|\mathbf{x}-\mathbf{y}|\,\tau^* + \boldsymbol{\lambda}.(\mathbf{x}-\mathbf{y})}{a_0\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}}, \tag{2·6}$$

we can write the volume integral, (2·3), in terms of the cross power spectral density tensor $H_{ijkl}(\boldsymbol{\eta}, \mathbf{k}, \omega)$:

$$\int \frac{\partial^4}{\partial \tau^4} P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}, \tau)\, \mathrm{d}\boldsymbol{\lambda} = (2\pi)^3 \int \omega^4 H_{ijkl}\left(\boldsymbol{\eta}, \frac{-\omega(\mathbf{x}-\mathbf{y})}{a_0\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}}, \omega\right) \exp\left\{\frac{\mathrm{i}\omega\,|\mathbf{x}-\mathbf{y}|\,\tau^*}{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})}\right\} \mathrm{d}\omega. \quad (2·7)$$

The time scale of $P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}, \sigma)$ is large (this was the object of introducing the $\boldsymbol{\lambda}$ variable), and consequently non-zero values of $H_{ijkl}(\boldsymbol{\eta}, \mathbf{k}, \omega)$ are confined to small values of frequency. Away from the singular point the wave-number vector

$$\mathbf{k} = \frac{-\omega(\mathbf{x}-\mathbf{y})}{a_0\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}}$$

is always confined to small values by this restriction on the range of frequency, ω, and must have a value close to zero. As a low-speed approximation we can treat the wave-number vector as zero to obtain Lighthill's approximation for low Mach number which neglects small differences in retarded time:

$$\int \frac{\partial^4}{\partial \tau^4} P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}, \tau)\, \mathrm{d}\boldsymbol{\lambda} \sim (2\pi)^3 \int \omega^4 H_{ijkl}(\boldsymbol{\eta}, 0, \omega) \exp\left\{\frac{\mathrm{i}\omega\,|\mathbf{x}-\mathbf{y}|\,\tau^*}{\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}}\right\} \mathrm{d}\omega$$
$$\sim \int \frac{\partial^4}{\partial \tau^4} P_{ijkl}\left(\boldsymbol{\eta}, \boldsymbol{\lambda}, \frac{\tau^*\,|\mathbf{x}-\mathbf{y}|}{\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}}\right) \mathrm{d}\boldsymbol{\lambda}. \quad (2·8)$$

Were the power at zero wave number very small, the presence of important octupole fields would be implied by the near-vanishing of the instantaneous volume integral. When a significant quadrupole strength density is present there will always be finite power at zero wave number and the suggested low-speed approximation will be valid provided the wave-number vector

$$\frac{\omega(\mathbf{x}-\mathbf{y})}{a_0\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}}$$

is small. But this may not be true at the higher values of ω/a_0. The restrictions on this approximation were fully discussed by Lighthill (1954) in his second paper.

At higher Mach numbers the situation is quite different, particularly in the regions of the singularities, and this can be illustrated by a further variable change.

The new frequency variable α will be introduced and defined by the relation

$$\left.\begin{aligned} \frac{\omega\,|\mathbf{x}-\mathbf{y}|}{\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}} &= \alpha, \\ \mathrm{d}\omega = \left|\frac{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right| \mathrm{d}\alpha \quad &\text{for positive elements.} \end{aligned}\right\} \quad (2·9)$$

Equation (2·7) is then

$$\int \frac{\partial^4}{\partial \tau^4} P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}, \tau)\, \mathrm{d}\boldsymbol{\lambda} = (2\pi)^3 \int \alpha^4 \left|\frac{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right|^5 H_{ijkl}\left(\boldsymbol{\eta}, \frac{-\alpha(\mathbf{x}-\mathbf{y})}{a_0\,|\mathbf{x}-\mathbf{y}|}, \frac{\alpha\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}}{|\mathbf{x}-\mathbf{y}|}\right) \mathrm{e}^{\mathrm{i}\alpha\tau^*}\, \mathrm{d}\alpha. \quad (2·10)$$

Near the singularity the frequency is confined to zero and the right-hand side becomes equal to

$$(2\pi)^3\left|\frac{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right|^5\int\alpha^4 H_{ijkl}\left(\boldsymbol{\eta},\ \frac{-\alpha(\mathbf{x}-\mathbf{y})}{a_0\,|\mathbf{x}-\mathbf{y}|},\ 0\right)e^{i\alpha\tau^*}\,d\alpha. \tag{2·11}$$

On replacing the main factor in the integrand by its Fourier transform

$$\frac{\alpha^4}{a_0^4}H_{ijkl}\left(\boldsymbol{\eta},\ \frac{-\alpha(\mathbf{x}-\mathbf{y})}{a_0\,|\mathbf{x}-\mathbf{y}|},\ 0\right)=\frac{1}{(2\pi)^4}\iint\frac{\partial^4}{\partial\lambda^4_{(\mathbf{x}-\mathbf{y})}}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\sigma)\exp\left\{\frac{i\alpha(\mathbf{x}-\mathbf{y}).\boldsymbol{\lambda}}{a_0\,|\mathbf{x}-\mathbf{y}|}\right\}d\sigma\,d\boldsymbol{\lambda}, \tag{2·12}$$

where $\partial/\partial\lambda_{(\mathbf{x}-\mathbf{y})}$ denotes differentiation with respect to the component of $\boldsymbol{\lambda}$ in the direction of emission, $(\mathbf{x}-\mathbf{y})$, equation (2·11) can be simplified to

$$\int\frac{\partial^4}{\partial\tau^4}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)\,d\boldsymbol{\lambda}$$
$$=\left|\frac{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right|^5\frac{a_0^4}{2\pi}\iint_v\frac{\partial^4}{\partial\lambda^4_{(\mathbf{x}-\mathbf{y})}}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\sigma)\,\delta\left\{\tau^*+\frac{\boldsymbol{\lambda}.(\mathbf{x}-\mathbf{y})}{a_0|\mathbf{x}-\mathbf{y}|}\right\}d\sigma\,d\boldsymbol{\lambda}_v. \tag{2·13}$$

The volume integral over $\boldsymbol{\lambda}$ can now be reduced to a surface integral by integrating in the direction of $(\mathbf{x}-\mathbf{y})$ giving

$$\int\frac{\partial^4}{\partial\tau^4}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)\,d\boldsymbol{\lambda}=\left|\frac{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right|^5 a_0^5\iint_s\frac{\partial^4}{\partial\lambda^4_{(\mathbf{x}-\mathbf{y})}}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda}_s,\sigma)\,d\sigma\,d\boldsymbol{\lambda}_s, \tag{2·14}$$

where $d\boldsymbol{\lambda}_s$ is an area element in a plane normal to the direction of Mach wave emission and defined by the condition

$$a_0\tau^*\,|\mathbf{x}-\mathbf{y}|+(\mathbf{x}-\mathbf{y}).\boldsymbol{\lambda}_s=0. \tag{2·15}$$

The acoustic radiation in the region of the singularity can then be deduced from the formula obtained by combining equations (2·14) and (1·29)

$$B(\mathbf{x},t,\tau^*)\sim\frac{1}{16\pi^2a_0^3}\iiint_s\frac{(x_i-y_i)\,(x_j-y_j)\,(x_k-y_k)\,(x_l-y_l)}{\big||\mathbf{x}-\mathbf{y}|+\mathbf{N}.(\mathbf{x}-\mathbf{y})\big|\,|\mathbf{x}-\mathbf{y}|^5}\frac{\partial^4}{\partial\lambda^4_{(\mathbf{x}-\mathbf{y})}}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda}_s,\sigma)\,d\sigma\,d\boldsymbol{\lambda}_s\,d\boldsymbol{\eta}. \tag{2·16}$$

This equation dealing with a Mach wave emission has quite a different form from the usual one applied to low-speed flows and does not involve the apparent infinity at points satisfying the condition

$$\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}=0.$$

It is an exact equation at this singularity giving, for the first time, a direct value to the height of the peak in the directional distribution so clearly indicated by the infinity in the low-speed analysis. It is interesting to note that, in accordance with the concept of sonically-convected quadrupoles reducing to simple sources, the retarded-time differences play no part at all in this equation, but perhaps the most interesting feature is its dependence on the turbulence time scale. At low speeds this is one of the most important factors determining acoustic efficiency as the intensity is proportional to the fourth power of frequency. Phillips (1960), however, found quite a different frequency dependence and showed that the Mach wave intensity was directly proportional to the turbulence time scale or eddy life-time. This direct proportionality is again predicted both here and by Lighthill (1961) and will result in replacing the famous U^8 law by U^3 around the regions of peak emission.

3. Intensity and frequency analysis of the sound field

Both the acoustic intensity and frequency spectrum can be obtained from the density autocorrelation function. The intensity $I(\mathbf{x}, t)$ is given by

$$I(\mathbf{x}, t) = a_0^3 \rho_0^{-1} B(\mathbf{x}, t, 0), \tag{3·1}$$

and the power spectral density of the sound field $W(\mathbf{x}, t, \gamma)$ by

$$W(\mathbf{x}, t, \gamma) = \frac{a_0^3}{2\pi} \rho_0^{-1} \int B(\mathbf{x}, t, \tau^*)\, \mathrm{e}^{-\mathrm{i}\gamma\tau^*}\, \mathrm{d}\tau^*. \tag{3·2}$$

For simplicity we can let the observer be at the origin of co-ordinates, i.e. $\mathbf{x} = (0, 0, 0)$ and write

$$\mathbf{M}.(\mathbf{x}-\mathbf{y}) = |\mathbf{x}-\mathbf{y}|\, M \cos\theta, \tag{3·3}$$

and

$$\mathbf{N}.(\mathbf{x}-\mathbf{y}) = |\mathbf{x}-\mathbf{y}|\, N \cos\phi. \tag{3·4}$$

At Mach numbers outside the range $(1 - M\cos\theta) \sim 0$ (and the extent of this range is illustrated by the example of § 5) the approximate form given by Lighthill, where retarded-time differences are neglected in the moving frame, is most revealing. The intensity can then be written in a form which illustrates both the effect of aircraft motion and eddy convection

$$I(\mathbf{x}, t) \sim \frac{1}{16\pi^2 \rho_0 a_0^5} \iint \frac{y_i y_j y_k y_l}{|\mathbf{y}|^6\, |1+N\cos\phi|\, |1-M\cos\theta|^5} \frac{\partial^4}{\partial\tau^4} P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}, 0)\, \mathrm{d}\boldsymbol{\lambda}_v\, \mathrm{d}\boldsymbol{\eta}. \tag{3·5}$$

At supersonic values of eddy convection Mach numbers the peak intensity will be around the angle $\cos\theta = M^{-1}$, where the above approximate result is singular. The intensity is then given the non-singular equation

$$I(\mathbf{x}, t)_{(1-M\cos\theta)=0} \sim \frac{1}{16\pi^2 \rho_0} \iiint_s \frac{y_i y_j y_k y_l}{|\mathbf{y}|^6\, |1+N\cos\phi|} \frac{\partial^4}{\partial\lambda^4_{(\mathbf{x}-\mathbf{y})}} P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}_s, \sigma)\, \mathrm{d}\sigma\, \mathrm{d}\boldsymbol{\lambda}_s\, \mathrm{d}\boldsymbol{\eta}. \tag{3·6}$$

In both cases the main effect of aircraft motion is contained in the $|1+N\cos\phi|$ factor, although velocity changes do affect the value of the correlation tensor $P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}, \sigma)$.

The frequency spectra of the radiation field in both these régimes are of interest, and especially the components of turbulent motion which generate particular frequencies of sound.

The power spectral density can be written by combining equations (3·2), (2·7) and (1·29)

$$W(\mathbf{x}, t, \gamma) \sim \frac{1}{4\rho_0 a_0^5} \iiint \frac{y_i y_j y_k y_l \omega^4}{|\mathbf{y}|^6\, |1+N\cos\phi|\, |1-M\cos\theta|^5} \times H_{ijkl}\left\{\boldsymbol{\eta}, -\frac{\omega(\mathbf{x}-\mathbf{y})\,(1-M\cos\theta)^{-1}}{a_0\,|\mathbf{x}-\mathbf{y}|}, \omega\right\} \exp\left\{-\mathrm{i}\tau^*\left(\gamma + \frac{\omega}{(1-M\cos\theta)}\right)\right\} \mathrm{d}\tau^*\, \mathrm{d}\omega\, \mathrm{d}\boldsymbol{\eta} \tag{3·7}$$

$$\sim \frac{\pi}{2\rho_0 a_0^5} \int \frac{y_i y_j y_k y_l \gamma^4}{|\mathbf{y}|^6\, |1+N\cos\phi|} H_{ijkl}\left\{\boldsymbol{\eta}, \frac{\gamma}{a_0} \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}, -\gamma(1-M\cos\theta)\right\} \mathrm{d}\boldsymbol{\eta}. \tag{3·8}$$

An interesting observation can be made at this stage regarding the components of the turbulence which generate particular parts of the sound field. The analysis presented here is perfectly valid for any value of the frame Mach number M, and it may be seen that the wave-number vector of the sound is precisely the same as the wave number of the turbulence which generated it. On the other hand, the frequency in the turbulence must be scaled by the Doppler factor $(1-M\cos\theta)$ to give the frequency of the emitted sound wave.

This point can be illustrated by examining a typical power spectral density function which might be observed in convected turbulence. Of course such functions have never been measured experimentally and one can only assume a likely form. Consider for the moment a fixed-frame analysis applied to convected turbulence. M and N are both put to zero. A power spectral density can be defined as a function of the wave-number component in the direction of emission, k_θ, and turbulence frequency, ω. The other two wave-number components are assumed zero to give the turbulence spectral function directly responsible for radiation in the chosen direction. The main power will be associated with velocity $-U_c\cos\theta$ as turbulence frequencies are mainly due to wave-number convection. Here U_c is the eddy convection velocity and θ, the emission angle, is assumed to be small. Figure 3 illustrates this situation for an idealized case.

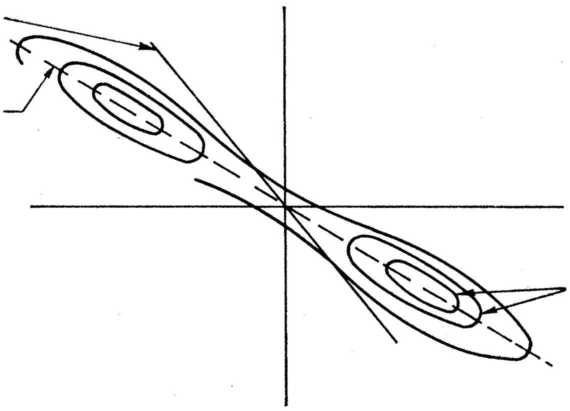

FIGURE 3. Sketch of a typical cross-section of the four-dimensional power function, $H_{ijkl}(\eta, \mathbf{k}, \omega)$, showing the sound-producing elements. (FIXED AXES.)

From this illustration it is clear that the wave numbers which can make a contribution to the radiation field are much smaller than the typical dominant wave number of the turbulence. It has been shown in §2 that the neglect of retarded-time differences restricts the analysis to zero wave number. Clearly such a step will not be justified in the fixed axes unless the power along the line defined by the sonic velocity, $-a_0$, is approximately equal to that on the frequency axis.

In a moving axis analysis where the time scales are maximized the main power is concentrated around the zero frequency line. In this case the sound is only generated by the components lying on the line defined by the velocity $-a_0(1-M\cos\theta)$. The moving-axis sketch of figure 4 shows that the power on the line responsible for sound production is approximately equal to the power at zero wave number. That is only so if the factor $(1-M\cos\theta)$ is not close to zero. Here then, retarded-time differences may be negligible and Lighthill's low-speed approximation fully justified.

If these sketches are typical of the power distribution in convected turbulence, they demonstrate how only the low wave-number components of the turbulence stress tensor

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generate sound. The high wave numbers could make a contribution if the moving-axis frequencies were sufficiently high as to allow significant power to fall on the velocity line responsible for radiation. As the singularity is approached, $(1-M\cos\theta)\to 0$, the velocity line becomes coincident with the wave-number axis and then all wave numbers contribute.

It is important to notice that the zero wave-number component of T_{ij} cannot radiate sound. Now in the low-speed approximation where retarded-time differences are neglected the analysis is restricted to zero wave number, for instantaneous volume integrals of non-zero components vanish. The acoustic power spectral density is then given by the approximate form

$$W(\mathbf{x},t,\gamma)\sim\frac{\pi}{2\rho_0 a_0^5}\int\frac{y_i y_j y_k y_l}{|\mathbf{y}|^6\,|1+N\cos\phi)|}\gamma^4 H_{ijkl}(\boldsymbol{\eta},0,-\gamma(1-M\cos\theta))\,\mathrm{d}\boldsymbol{\eta}. \tag{3·9}$$

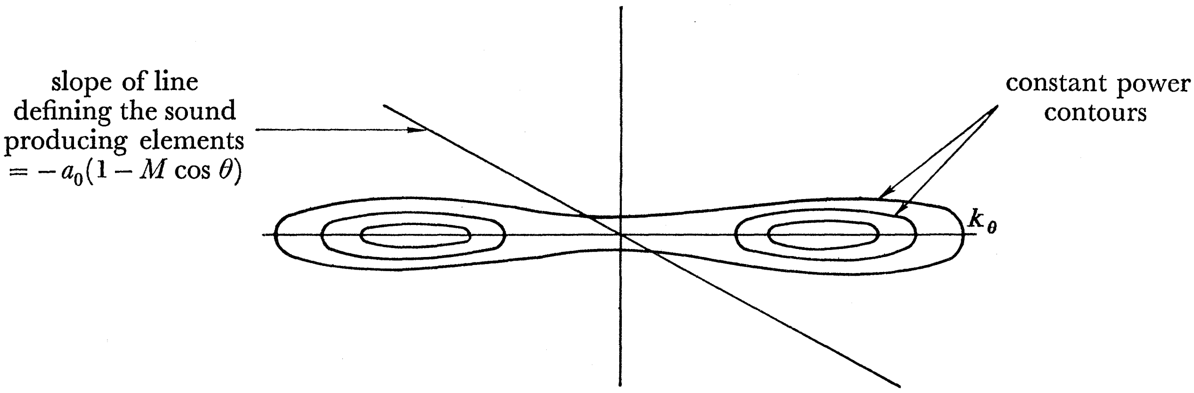

FIGURE 4. Sketch of a typical cross-section of the four-dimensional power function, $H_{ijkl}(\boldsymbol{\eta},\mathbf{k},\omega)$, showing the sound-producing elements. (MOVING AXES.)

It is very important to note the significance of this result. It does not imply that all frequencies in the sound field are generated by turbulence at zero wave number—indeed such turbulence radiates no sound. The result is essentially an approximation which may be useful in estimating magnitude. Its meaning is that the power in T_{ij} at the desired wave number of $\gamma(\mathbf{x}-\mathbf{y})/a_0\,|\mathbf{x}-\mathbf{y}|$, which is assumed small, is approximately equal to the power at $\mathbf{k}=0$. This is because a Fourier integral of a function with small wave number approximates to the volume integral of the function over all space, leading to Lighthill's eddy volume formulation (Lighthill 1954). Such an approximation has obvious limitations when it is used as a basis of further analysis as the step focuses attention on parts of the turbulence which cannot produce sound.

It should be noticed that power at a given frequency and zero wave number is completely independent of the frame of reference. This suggests the possibility of providing data from filtered observations, but the filter must be set at the Doppler-shifted frequency although the experiments might be performed in the same reference frame as the observer. Application of a differentiating circuit on to a stationary probe is also admissible in this case although care should be taken to ensure that the measured signal is not predominantly of a type that contributes nothing to the integral.

At the singularity where the emission of sound is much more of a Mach wave phenomenon the situation is again quite different. The radiation spectrum can be calculated from equations (3·2), (2·11) and (1·29) or more simply by inserting the singular conditions in equation (3·8):

$$W(\mathbf{x}, t, \gamma)_{(1-M\cos\theta)=0} \sim \frac{\pi}{2\rho_0 a_0^5} \int \frac{y_i y_j y_k y_l}{|\mathbf{y}|^6 \, |1+N\cos\phi|} \gamma^4 H_{ijkl}\left(\boldsymbol{\eta}, \frac{\gamma(\mathbf{x}-\mathbf{y})}{a_0 |\mathbf{x}-\mathbf{y}|}, 0\right) \mathrm{d}\boldsymbol{\eta}, \quad (3·10)$$

showing again that the wave number in the turbulence and the sound produced are identical. But in this case the zero-frequency components generate the sound (these are integrals of correlations with respect to time which can be expressed in terms of a sort of eddy persistence time)—contrasting vividly with the régime discussed above but again analogous with the high-speed case examined by Phillips (1960).

Mach wave emission of this type is expected to correspond to the directional peaks whereever supersonic convection is present. At these conditions all wave numbers will contribute to the radiation and a wider range of acoustic frequencies might be expected. However, if the example of § 5 is at all realistic, very high acoustic efficiences will be present at supersonic speeds with most of the radiation being of the Mach wave type. At these high efficiencies considerable radiation damping of the turbulence will be expected, and the high wavenumber turbulence being the most efficient radiator will be the first to be affected. Radiation damping will then reduce the wave-number range and hence the radiation frequencies. This perhaps is the effect noted by Lilley (1958) in Lassiter & Heitkotter's (1954) work where peak frequencies in the rocket noise-field were much lower than those generated by subsonic jets.

It might be postulated that where turbulence frequencies determine the acoustic spectrum, the spectra might collapse on a Strouhal number basis, but for the Mach wave radiation the spectrum would tend to a function independent of speed.

4. Dimensional analysis and asymptotic forms

One of the important results in Lighthill's (1952) theory is the prediction that acoustic strength increases with the eighth power of velocity. This result, derived from a dimensional analysis of the equations governing radiation, has provided the basis for co-ordinating experimental results. A dimensional analysis for the condition of Mach wave radiation is given below. This could serve a similar purpose in providing a basis for co-ordinating the peak noise levels in high-speed jets and rocket engines—provided, of course, that the noise orginates from turbulence.

Another important aspect which can be studied by dimensional reasoning concerns the value of the acoustic efficiency at high speeds. Lighthill (1952) showed it to increase with the fifth power of velocity at low speed but his theory took no account of the back-reaction of sound on its turbulence source. If the efficiency continues its increase, at high speeds the turbulence structure must be affected considerably and no theory would be complete which ignored its effect. Phillips's (1960) work, which showed the efficiency to decrease with velocity at high speeds, suggests that the acoustic efficiency is always small and that added refinements accounting for turbulence damping are not called for. His prediction that an efficiency maximum might be observed at some supersonic velocity is qualitatively confirmed by Chobotov & Powell (1957) who give evidence that it approaches a constant in the speed range 2000 to 5000 ft/s.

Dimensional reasoning applied to the present theory shows the efficiency to asymptote to a constant after passing through a maximum, and in this respect differs from Phillips's result. This conclusion is based on the assumption that the dimensional form of the stress tensor T_{ij} is $\bar{\rho}U^2$, $\bar{\rho}$ being the mean density and U a typical velocity in the turbulence. The absence of experimental evidence on this point makes it difficult to be more precise. Lighthill (1954) has shown that the fluctuating momentum flux, $\rho u_i u_j$, is likely to be the most important term in the absence of large temperature changes, but to attempt a more refined specification than $\bar{\rho}U^2$ at the high Mach numbers would, at present, be a case of pure conjecture.

For an aircraft in flight the flow variables which might affect the sound are: U_0 the aircraft forward speed, U_1 the jet exit velocity relative to the aircraft, ρ the mean density in the jet flow, ρ_0 the undisturbed density of the surrounding medium, D a typical length scale such as jet diameter and $|\mathbf{y}|$ the distance from the aircraft to the observer at the time sound was emitted.

The correlation tensor $P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}, \sigma)$ is dimensionally similar to $\bar{\rho}^2(U_1-U_0)^4$ and the differentiation with respect to time can be done by dividing by a typical time scale, $D(U_1-U_0)^{-1}$,

$$\frac{\partial^4}{\partial \sigma^4} P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}, \sigma) \sim \frac{\bar{\rho}^2}{D^4}(U_1-U_0)^8. \tag{4·1}$$

An integration over $\boldsymbol{\lambda}$ is represented by multiplying the integrand by a typical volumetric scale, D^3, while an $\boldsymbol{\eta}$ integration is performed by multiplying this by the mixing region volume which varies like $D^3U_1/(U_1-U_0)$ (Squire & Trouncer 1944).

In regions away from the singularity at $|\mathbf{x}-\mathbf{y}| = \mathbf{M}.(\mathbf{x}-\mathbf{y})$, the equation for the acoustic intensity can be written in the simplified form:

$$I(0,t) \sim \frac{1}{16\pi^2\rho_0 a_0^5}\iint \frac{y_i y_j y_k y_l}{|\mathbf{y}|^6\,|1+N\cos\phi|}\,|1-M\cos\theta|^{-5}\frac{\partial^4}{\partial\sigma^4}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda},\sigma=0)\,\mathrm{d}\lambda_v\,\mathrm{d}\boldsymbol{\eta}, \tag{4·2}$$

where retarded-time differences are assumed small compared to the turbulence time scale.

This equation can then be analyzed dimensionally to give the low-speed dependence which applies to a moving jet,

$$I(0,t) \sim \frac{\rho^2}{\rho_0}\frac{(U_1-U_0)^7\,U_1}{a_0^5}\frac{D^2}{|\mathbf{y}|^2}\,|1-M\cos\theta|^{-5}\,|1+N\cos\phi|^{-1}. \tag{4·3}$$

This result has been obtained previously (Ffowcs Williams 1960) and is only significantly different from Lighthill's form (1954) in that proper account is taken of the fact that eddies move relative to the moving aircraft.

At high speeds away from the singularities where $M\cos\theta \gg 1$, we obtain the asymptotic high-speed form which shows the intensity to increase proportionally to jet power,

$$I(0,t)_{|M\cos\theta|\gg 1} \sim \frac{\bar{\rho}^2}{\rho_0}\frac{D^2}{|\mathbf{y}|^2}\,U_1^3|\cos\theta|^{-5}\,|1+N\cos\phi|^{-1}. \tag{4·4}$$

The dimensional form at and near the singularity where Mach wave emission is predominant can be obtained from an analysis of equation (2·16), here stated in the same simplified notation

$$I(0,t)_{M\cos\theta=1} \sim \frac{1}{16\pi^2\rho_0}\iiint_s \frac{y_i y_j y_k y_l}{|\mathbf{y}|^6\,|1+N\cos\phi|}\,\frac{\partial^4}{\partial\lambda^4_{(\mathbf{x}-\mathbf{y})}}P_{ijkl}(\boldsymbol{\eta},\boldsymbol{\lambda}_s,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\boldsymbol{\lambda}_s\,\mathrm{d}\boldsymbol{\eta}. \tag{4·5}$$

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The expression $(\partial^4/\partial\lambda^4_{(\mathbf{x}-\mathbf{y})})P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}_s, \sigma)$ is likely to be a function of the angle of emission, θ, if the space scales of the turbulence are not all equal. Since θ is a function of M for the singular condition this introduces a further Mach number dependence the effect of which is difficult to assess in general, but can be seen quite clearly from the example which follows.

Dimensionally the operation of differentiation with respect to $\boldsymbol{\lambda}$ is effected by dividing by the typical scale D and the integration over time, σ, is represented by multiplying the integrand by a typical time scale $D/(U_1 - U_0)$.

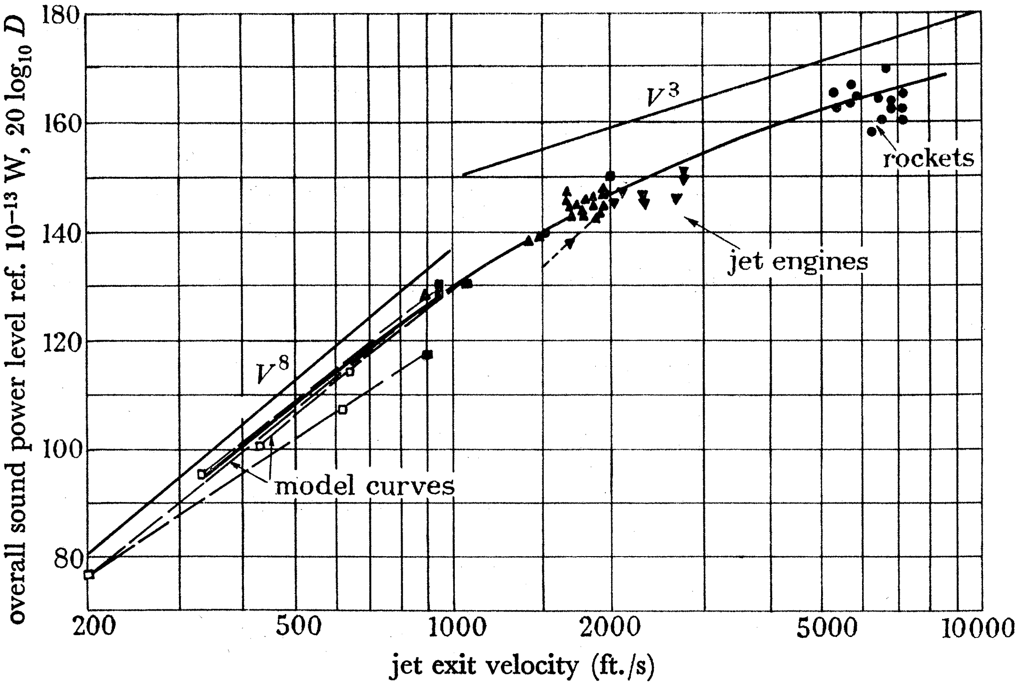

FIGURE 5. Variation of acoustic power levels from Chobotov & Powell (1957). ●, Rocket; ▼, turbojet (afterburning); ▲, turbojet (military power); ■, exit velocity > M = 0·8; □, air model (exit velocity < M = 0·8). D is the exit diameter in inches.

The dimensional form of the intensity equation at the singularity which reveals the nature of the Mach wave emission can then be specified

$$I(0,t)_{M\cos\theta=1} \sim \frac{\bar{\rho}^2}{\rho_0}\frac{D^2}{|\mathbf{y}|^2}(U_1 - U_0)^2 U_1 \,|1 + N\cos\phi|^{-1}. \tag{4·6}$$

At high jet exit velocities when $U_1 \gg U_0$ this has a form very similar to the one developed for quadrupole radiation at very high speeds,

$$I(0,t)_{M\cos\theta=1} \sim \frac{\bar{\rho}^2}{\rho_0}\frac{D^2}{|\mathbf{y}|^2} U_1^3 \,|1 + N\cos\phi|^{-1}. \tag{4·7}$$

The total rate of energy supply to the jet exhaust is $\rho U_1^3 D^2$ and the ratio of acoustic power output to the supply of power represents an acoustic efficiency which we shall call η. By the above rather crude dimensional reasoning the acoustic efficiency is shown to tend to a constant value at high speed and the example following will suggest that a maximum value exists. The asymptotic constant is easily shown to depend only on the density ratio and aircraft Mach number,

$$\eta_{U_1\to\infty} \sim (\bar{\rho}/\rho_0) f(N). \tag{4·8}$$

The inference that the radiated power asymptotes to a dependence on the velocity cubed finds some confirmation in the evidence of Chobotov & Powell (1957). Figure 5 is reproduced from their report and provides a very crude qualitative confirmation of the predicted dimensional trend, but in assessing the value of this confirmation it must be remembered that the dimensional arguments are very crude.

5. A particular example

To illustrate the nature of acoustic emission over a wide range of convection velocities an example is considered where all elements of the correlation tensor are similar in form, i.e.

$$P_{ijkl}(\boldsymbol{\eta}, \boldsymbol{\lambda}, \tau) \equiv A_{ijkl}(\boldsymbol{\eta})\, \bar{P}(\boldsymbol{\lambda}, \tau). \tag{5·1}$$

With this type of correlation function the intensity equation (1·29) reduces to

$$I(0,t) \sim \frac{1}{16\pi^2 \rho_0 a_0^5} \int\int \frac{y_i y_j y_k y_l A_{ijkl}(\boldsymbol{\eta})}{|\mathbf{y}|^6\, |1+N\cos\phi|\, |1-M\cos\theta|^5} \frac{\partial^4}{\partial\tau^4} \bar{P}(\boldsymbol{\lambda},\tau)\, \mathrm{d}\boldsymbol{\lambda}\, \mathrm{d}\boldsymbol{\eta}, \tag{5·2}$$

where the simplified notation of § 3 is used and the jet axis lies in the 1 direction in the plane $y_3 = 0$.

The effect of eddy convection is now entirely contained in the integral

$$\int |1-M\cos\theta|^{-5} \frac{\partial^4}{\partial\tau^4} \bar{P}(\boldsymbol{\lambda},\tau)\, \mathrm{d}\boldsymbol{\lambda}, \tag{5·3}$$

and its evaluation over a range of Mach numbers, M, will illustrate the augmentation due to convection.

The correlation function chosen for the example is

$$\bar{P}(\boldsymbol{\lambda},\tau) = \exp -\{a_1^2\lambda_1^2 + a_2^2\lambda_2^2 + a_3^2\lambda_3^2 + \beta^2 M^2 \tau^2\}. \tag{5·4}$$

This represents a hypothetical but possible form where turbulent scales are constant while frequency varies directly with velocity.

In the exact integral the retarded time τ is completely dependent on the separation vector, $\boldsymbol{\lambda}$, and in this particular situation on the two components λ_1 and λ_2, parallel and perpendicular to the jet axis respectively,

$$\tau = \frac{\lambda_1\cos\theta + \lambda_2\sin\theta}{a_0(1-M\cos\theta)}. \tag{5·5}$$

The retarded-time integral can then be evaluated directly to give

$$\int |1-M\cos\theta|^{-5} \frac{\partial^4}{\partial\tau^4} \bar{P}(\boldsymbol{\lambda},\tau)\, \mathrm{d}\boldsymbol{\lambda} = 12\beta^4 M^4 \frac{\pi^{\frac{3}{2}}}{a_1 a_2 a_3} \left\{ (1-M\cos\theta)^2 + \frac{\beta^2 M^2 \cos^2\theta}{a_1^2 a_0^2} \left(1 + \frac{a_1^2}{a_2^2}\tan^2\theta\right) \right\}^{-\frac{5}{2}}. \tag{5·6}$$

Neglect of the retarded-time differences would yield the approximate low-speed result

$$12\beta^4 M^4 \frac{\pi^{\frac{3}{2}}}{a_1 a_2 a_3} |1-M\cos\theta|^{-5}. \tag{5·7}$$

A comparison of the exact form with the approximate result will show the range over which the low-speed approximation is valid. In this particular case the approximation is shown to be good over a wide range of convection velocities as can be seen in figure 7.

In the more accurate result there appear to be two important ratios of turbulence scales which determine the main features of the Mach number dependence. The first is the ratio of time to longitudinal scales defined by

$$b_1 = \beta/a_1 a_0, \tag{5·8}$$

and the second the ratio of longitudinal to lateral scales

$$b_2 = a_1/a_2. \tag{5·9}$$

In model jets the ratio b_1 is found to be approximately $\frac{1}{5}$ at subsonic velocities while b_2 is near $\frac{1}{3}$. No evidence is available at present on whether or not they may be regarded as constant throughout the speed range but this will be assumed in the present example.

The condition which Lighthill (1954) gave on the validity of the moving-axes approximation was that frequencies should be sufficiently low. This condition is shown explicitly by the important role b_1 plays in determining the range over which retarded-time differences can be neglected.

In terms of these ratios the retarded-time integral is

$$12\beta^4 M^4 \frac{\pi^{\frac{3}{2}}}{a_1 a_2 a_3}\{(1-M\cos\theta)^2 + b_1^2 M^2(\cos^2\theta + b_2^2\sin^2\theta)\}^{-\frac{5}{2}}, \tag{5·10}$$

and retarded-time differences are negligible provided the observer is well away from the singular regions, i.e.

$$(1-M\cos\theta)^2 \gg b_1^2 M^2(\cos^2\theta + b_2^2\sin^2\theta). \tag{5·11}$$

The result then reduces to precisely the form found by the approximate method, equation (5·7).

Near the Mach angle however, the integral, (5·6), can be written

$$\frac{12\beta^4 M^4 \pi^{\frac{3}{2}}}{a_1 a_2 a_3 b_1^5}\{1 + b_2^2[M^2-1]\}^{-\frac{5}{2}}, \tag{5·12}$$

where the ratio b_1 plays a most important part in determining the strength of the Mach wave emission. The asymptotic form for high Mach numbers then reduces to

$$\frac{12\pi^{\frac{3}{2}} a_2^5 a_0^5}{\beta M a_1 a_2 a_3}, \tag{5·13}$$

which shows a dependence on the inverse frequency, $(\beta M)^{-1}$, and the fifth power of a typical wave number in the transverse direction, a_2^5.

The main effect of convection is illustrated by the

$$\{(1-M\cos\theta)^2 + b_1^2 M^2(\cos^2\theta + b_2^2\sin^2\theta)\}^{-\frac{5}{2}} \tag{5·14}$$

factor in (5·10), and this is the total augmentation due to eddy motion. The quadrupole strength is increasing with the eighth power of velocity in addition to this,

$$A_{ijkl}(\boldsymbol{\eta}) \sim U^4, \quad \beta^4 M^4 \sim U^4$$

and these two effects account for the entire velocity dependence.

The augmentation factor, (5·14), is shown in figures 6, 7 and 8 for a range of angular positions and convection Mach numbers. These figures bear a limited similarity to the

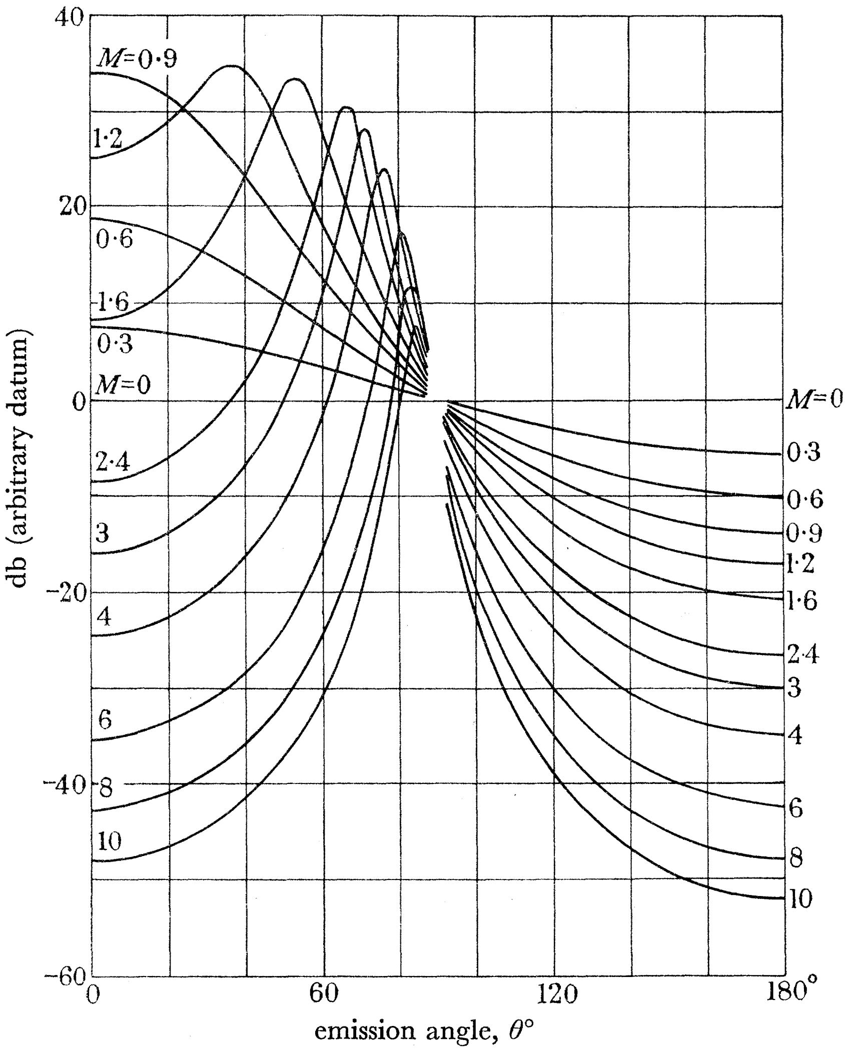

FIGURE 6. Directionality imposed by quadrupole convection at Mach number M.

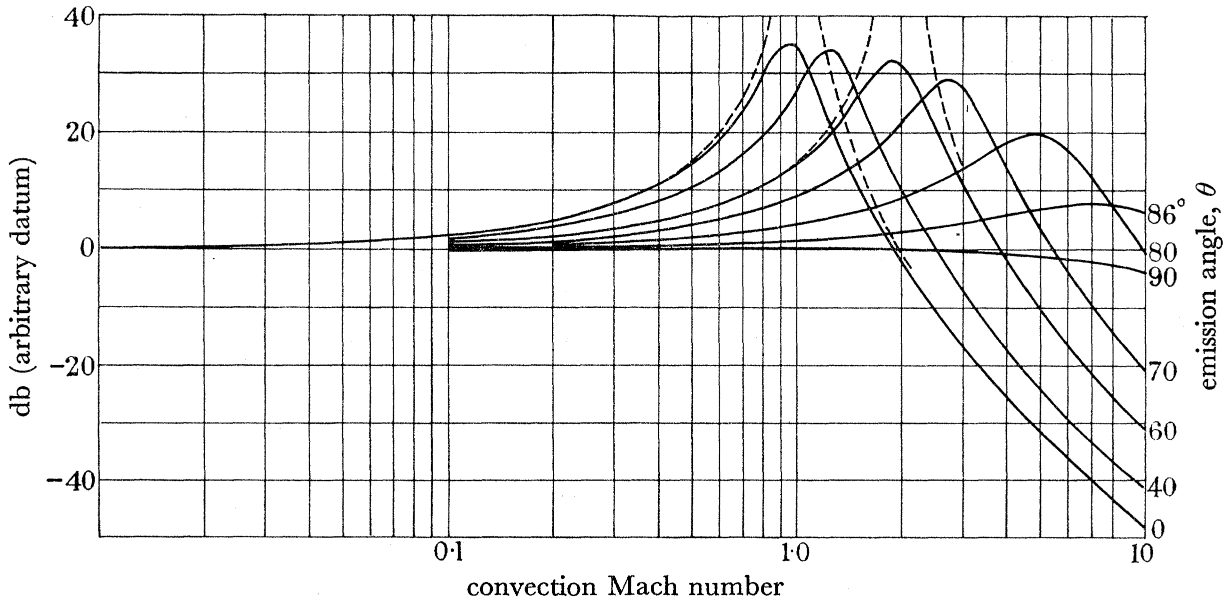

FIGURE 7. Amplification due to quadrupole convection at uniform Mach number. - - - -, Low-speed approximation neglecting retarded time; ————, theoretical example.

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illustration given by Ribner (1959). There he evaluated an idealized fixed-axis retarded-time integral to indicate a convection effect.

Figure 9 represents the predicted intensity variation with Mach numbers for a jet of unit power where the basic dependence of figure 7 has been augmented by the term U^5.

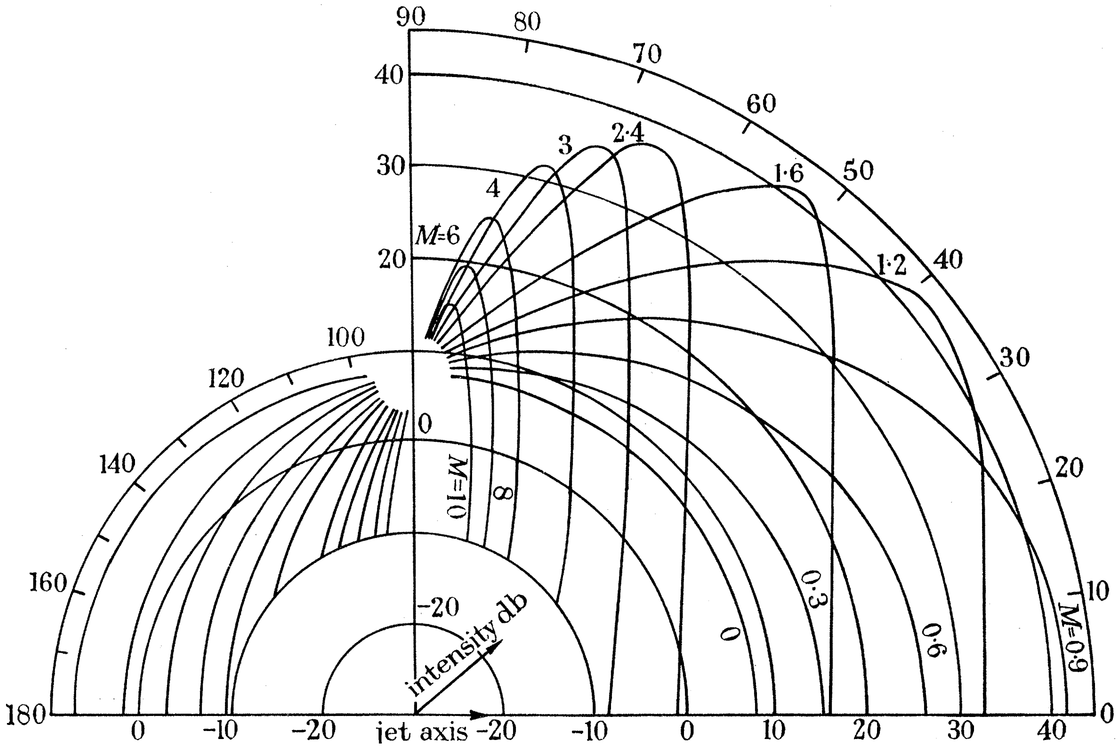

FIGURE 8. Directionality imposed by quadrupole convection at Mach number, M.

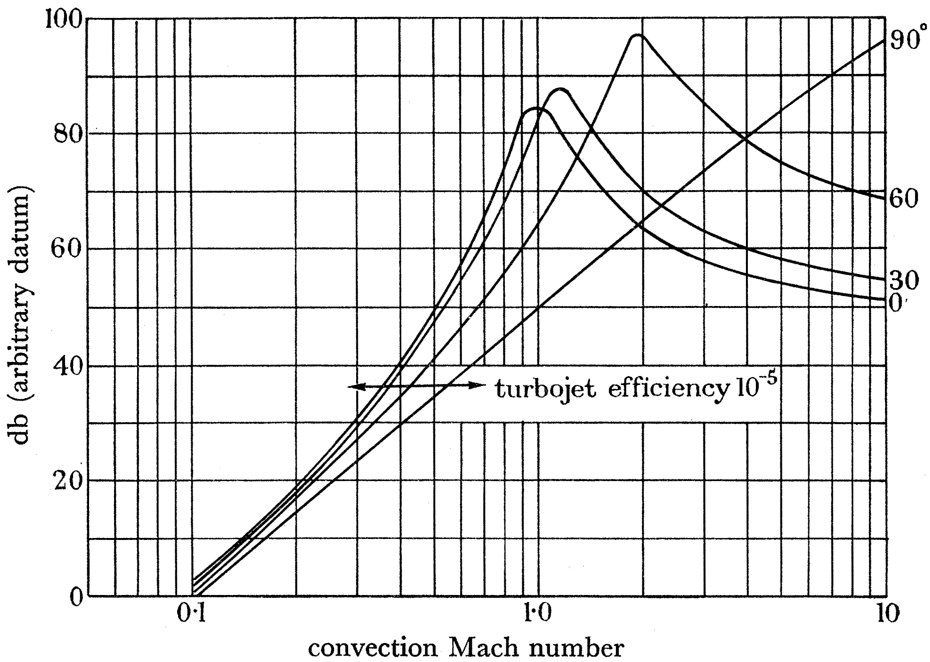

FIGURE 9. Theoretical variation of intensity per unit jet power with convection Mach number.

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The augmentation of the total power output by convection is shown in figure 10 for the three types of quadrupoles obtained by axis re-orientation. Figure 11 shows these curves augmented by the U^5 term remaining in the dimensional breakdown of the power equation.

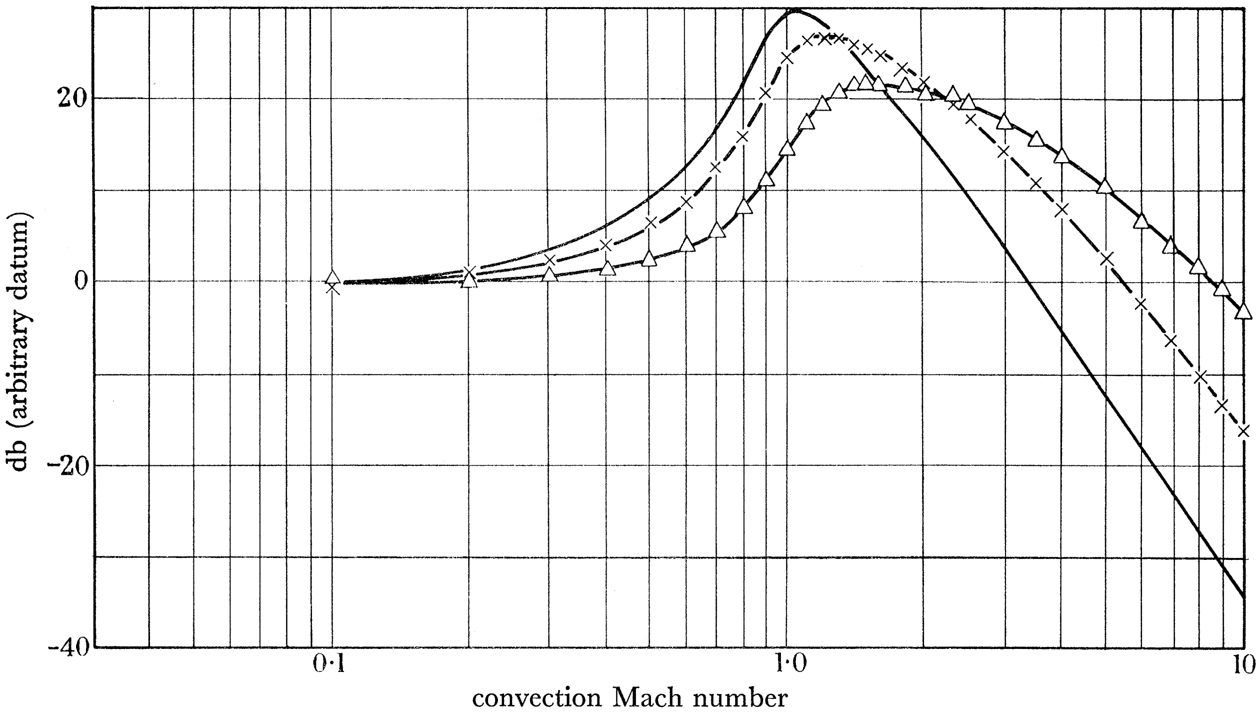

FIGURE 10. Augmentation of overall power output by quadrupole convection. ———, Both axes in jet direction (u_1u_1); –△–△–△–△–, both axes inclined perpendicular to jet direction (u_2u_2); –×–×–×–×–, one axis parallel, one axis perpendicular to jet direction (u_1u_2).

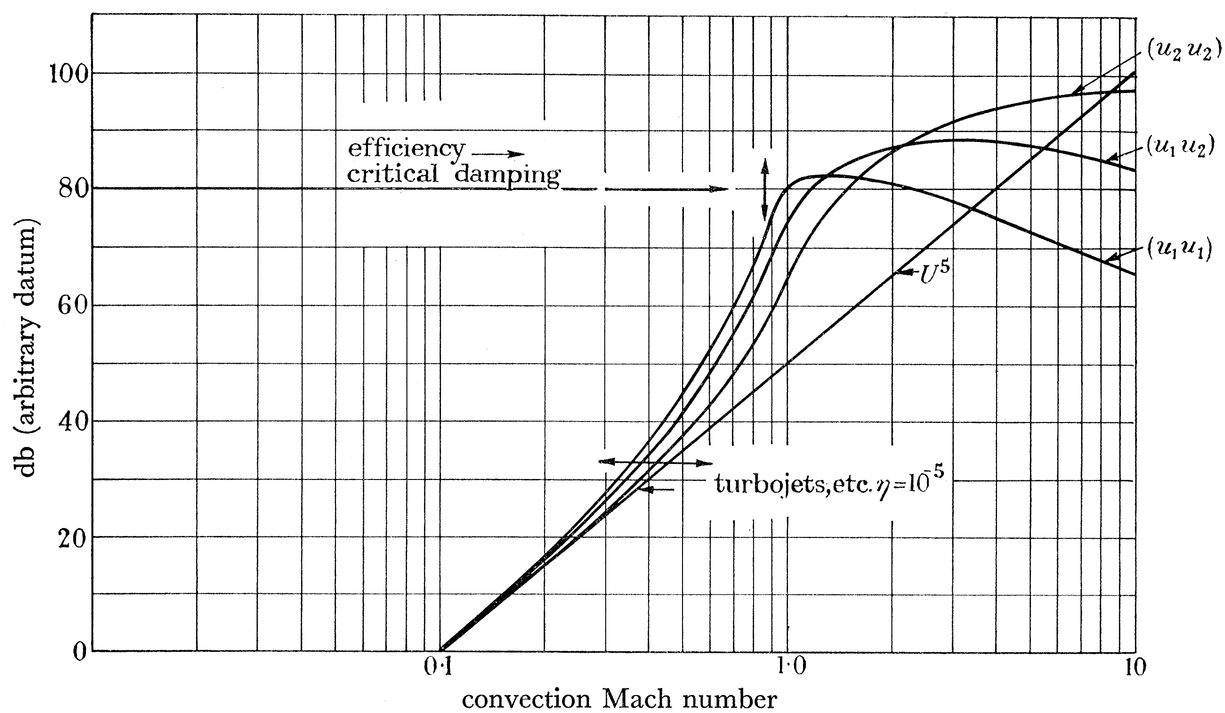

FIGURE 11. Variation of acoustic efficiency with convection Mach number.

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These curves then show the acoustic efficiency associated with each quadrupole type as it is convected through the otherwise stationary atmosphere. Although the dimensional analysis shows the efficiency to rise with U^5 at low speeds and asymptotes to a constant at high speeds, the example shows clearly a maximum value in the intermediate speed range.

Conclusion

At low speeds the present theory, being based on Lighthill's classic work, offers very little that is new apart from the modification needed to account for aircraft motion. Again acoustic frequencies are identified with those in the turbulence at values modified by the Doppler factor $(1-M\cos\theta)$. Admittedly a point of doubtful significance at low speeds is the identification of the acoustic wave number with that of the turbulence which produced it (see also Kraichnan (1953) and Mawardi (1955)), while it is well known that the radiated sound is usually at a very low wave number when compared to those typically found in turbulence. Of course, since the turbulence stress tensor depends on velocity quadratically, part of the low wave-number components arise from the sum and difference tones of the velocity fluctuations, i.e. the so-called 'turbulence-turbulence' contribution. Then the velocity fluctuations responsible for the sound generation may be of higher wave number. For example, in Proudman's (1952) analysis of isotropic turbulence it is the high wave-number velocity components which have the highest acoustic efficiency but these give rise to sound waves of comparatively low wave number. In heavily sheared jet mixing regions where $\partial^2/\partial t^2 T_{ij}$ can be approximated by $\bar{\tau}(\partial p/\partial t)$, $\bar{\tau}$ being a large mean shear and p the static pressure (see Lighthill 1954), there is no difference tone phenomenon and the wave number of sound is identical to that of p which produced it. It is most important to remember that the low wave-number spectral component refers not to energy in very large eddies but to an integral correlation function whose value may be obtained from a relatively small-scale correlation survey. The need to establish the low wave-number power spectral density in no way implies that extremely large-scale features of the turbulence must be studied in detail, a task made difficult by the near vanishing of the statistical correlations. Nor does it imply that these small correlations at large separations play a significant part in determining the power level, for they would only become significant if the volume integral were a near-vanishing quantity and the contributions from the small correlation regions were a significant fraction of the whole. But in that event a higher-order source system would be dominant and the analysis could be referred to the more complex sources to state the relevant equations in terms of a function which could be measured more easily by a localized small-scale survey.

At high speeds the intensity peaks heralded by the singularities of the approximate theory are given a finite value and are associated with a Mach wave emission similar in some respects to the phenomenon studied by Phillips (1960). Crude dimensional reasoning predicts the intensity of these peaks to vary with velocity cubed as does the limiting high-speed case of the more classical quadrupole emission. This high-speed velocity dependence is not the one predicted by Phillips (1960) in rather a different but analogous situation, although some detailed features show the two theories to bear certain similarities and both rely on what is essentially the same dimensional argument. The theories do, however, deal with quite different flow models with one concentrating on a two-dimensional system with a particular distribution of mean velocity and the other on a three-dimensional model demonstrating the effect of one typical convection velocity. In the absence of experimental data regarding the high speed form of the turbulence stress tensor, the dimensional arguments have been based on the assumption that T_{ij} is dimensionally similar to $\bar{\rho}U^2$. Although this may be a fair assumption up to low supersonic speeds it may be quite unacceptable as a basis for estimating the asymptotically high-speed trends. At very high speeds it is possibly more revealing, and certainly more correct, to state the results in the form:

$$I(0,t)_{|M\cos\theta|\gg 1} \sim \frac{1}{\rho_0}\frac{D^2}{|\mathbf{y}|^2}U^{-1}|\cos\theta|^{-5}|1+N\cos\phi|^{-1}\overline{T^2},$$

$$I(0,t)_{M\cos\theta=1} \sim \frac{1}{\rho_0}\frac{D^2}{|\mathbf{y}|^2}U^{-1}|1+N\cos\phi|^{-1}\overline{T^2},$$

where the dimensional form of $\overline{T^2}$ is left unspecified.

The main high-speed effects that the present theory predicts are the dependence of acoustic intensity on velocity cubed, the pronounced directionality of the acoustic field and a tendency for the acoustic spectrum at the position of peak emission to include high frequencies yet be independent of jet velocity. Experimental evidence at supersonic speed is scarce and difficult to interpret, often being associated with complex flow structures of the rocket exhaust type. To seek a detailed comparison of theoretical prediction with experimental observation in the absence of a reliable model of the mixing flow would be futile but more general features can be used in a qualitative manner with considerable success.

The extremely directional field radiated at $\cos\theta = M^{-1}$ has been observed in the noise of high-speed jets and in the radiation from supersonic turbulent boundary layers (Williams & Stevenson 1957; Laufer 1961). A shadowgraph picture taken by Ricketson is reproduced in figure 12, plate 5. This shows quite clearly the pronounced directionality of waves attached to eddies which evidently move at half the jet exit velocity. Figure 16 of the Bakerian lecture (Lighthill 1961) shows the same phenomenon, but associated with larger-scale features of the flow; again the waves are generated by eddies moving close to half the jet exit speed. That this highly directional field is present in the radiation from turbulent boundary layers is now well established. Particularly good shadowgraph pictures, figure 13, plate 6, are given by James (1958) (reproduced here by kind permission of the N.A.S.A.—Ames Research Centre where they were obtained) and the detailed study of the velocity perturbations in a supersonic wind tunnel conducted by Laufer (1961) established the existence of a highly directional wave field and gave for the first time a direct measure of the Mach wave strength. The intensity of this emission Laufer finds proportional to M^4, but the influence of the boundary is not at all clear. The eddy velocity is close to the average shear layer mean velocity while the wall pressure pattern is known to move with a speed nearly half as much again (Kistler & Laufer 1960). If the waves were generated by wall pressure stresses they would appear attached to eddies moving with this higher convection velocity.

A dimensional reasoning of the type conducted in this paper, if applied to the radiation field of wall pressure stresses at the Mach wave condition, would predict the intensity to increase with the sixth power of Mach number. The measured M^4 dependence found by Laufer (1961) together with an evident source speed considerably different to that of the

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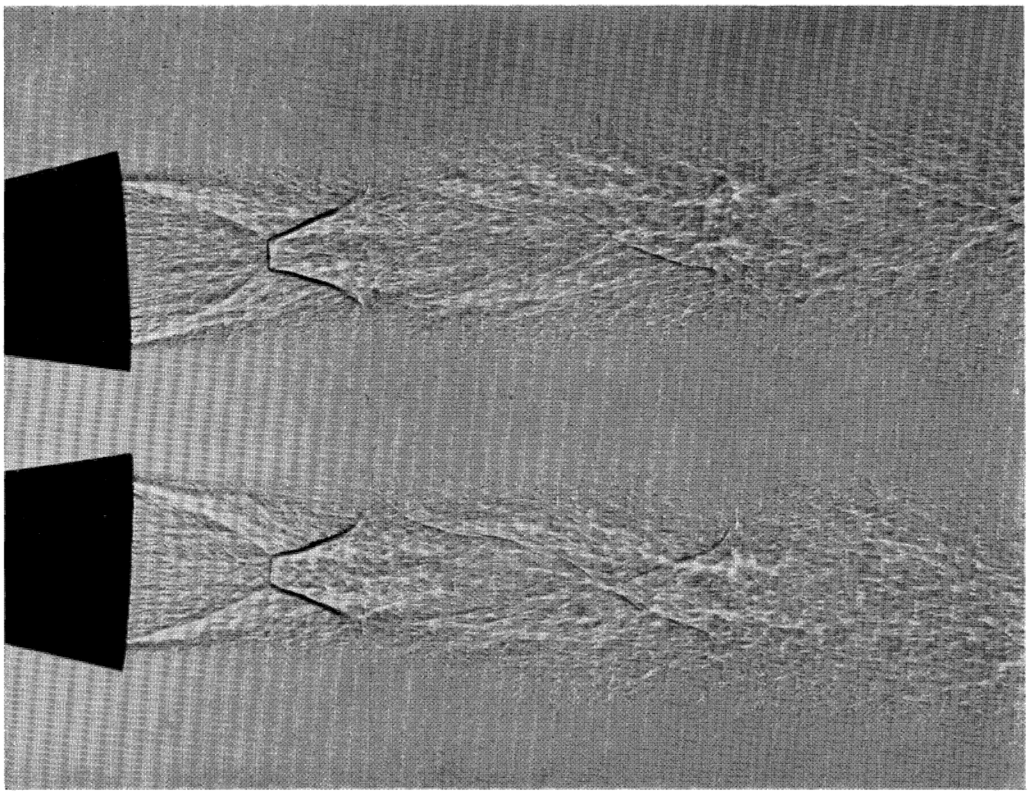

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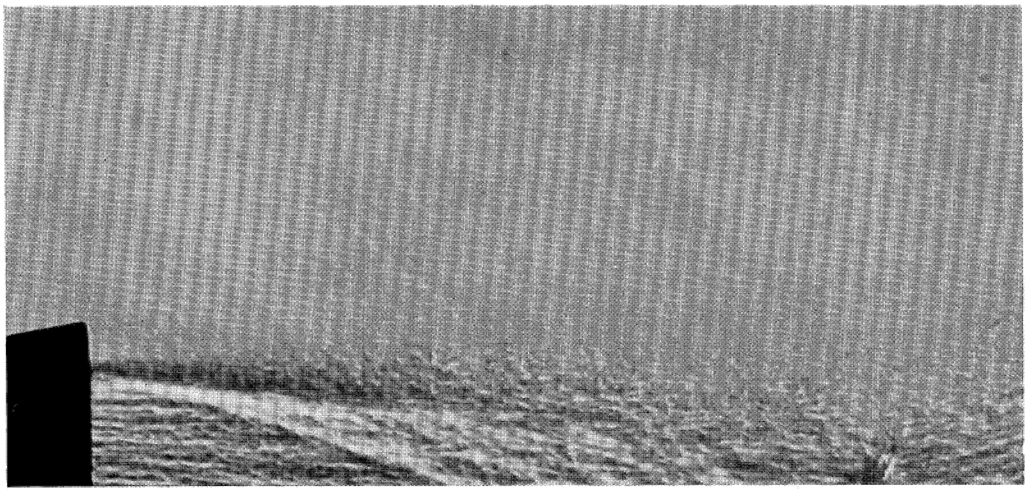

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FIGURE 12. Shadowgraph pictures illustrating Mach wave radiation generated by the exhaust of high-speed jets. (Crown copyright is reserved and this photograph is published by permission of Her Majesty's Stationery Office.)

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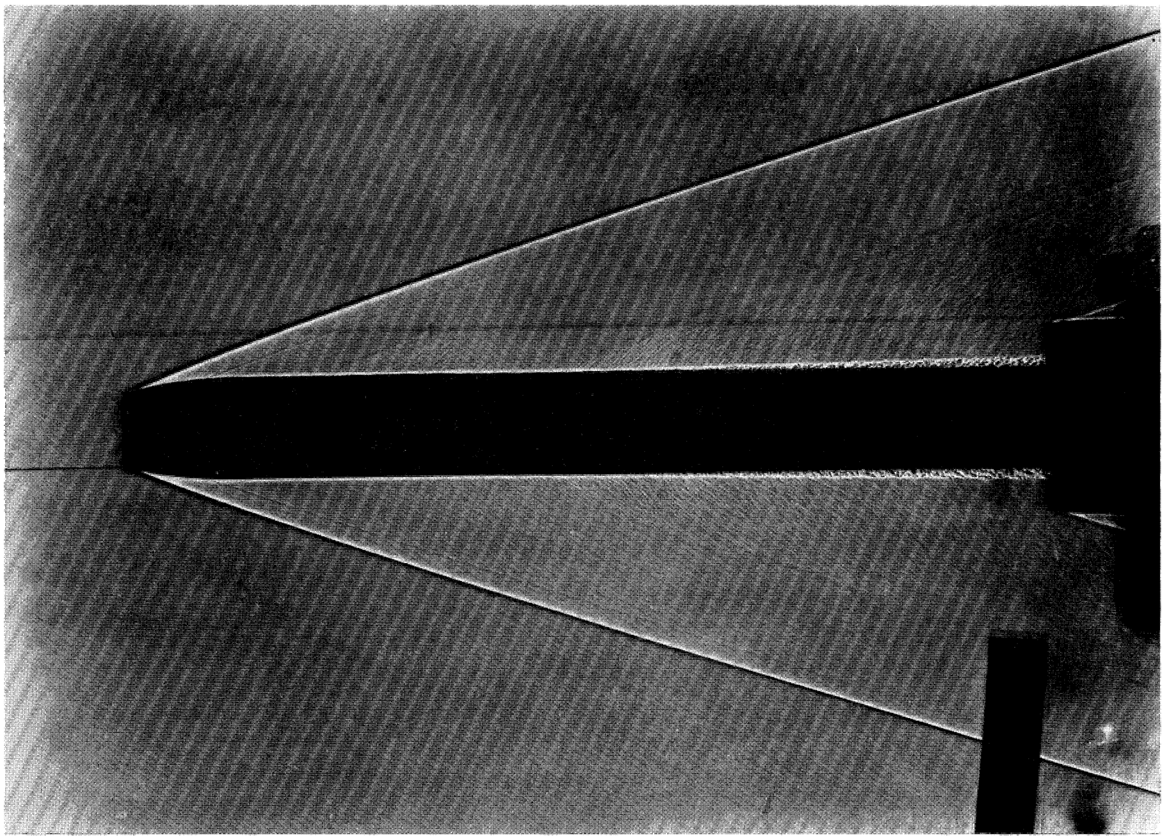

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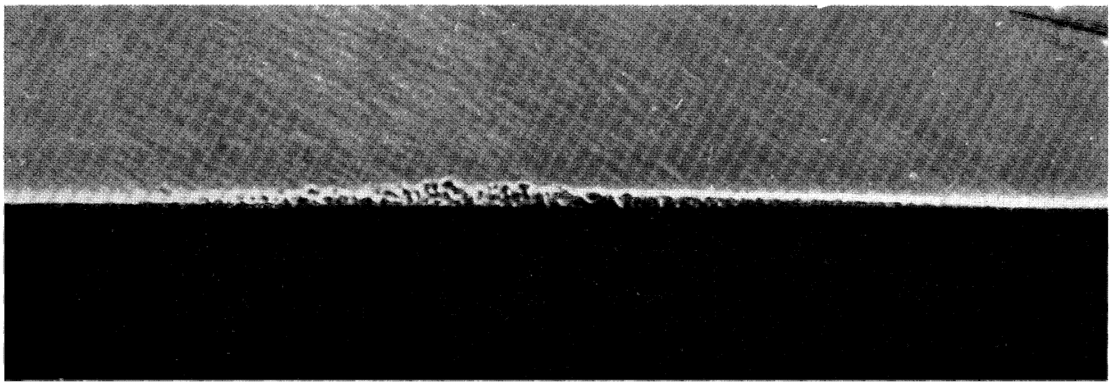

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FIGURE 13. Shadowgraph pictures illustrating Mach wave radiation generated by a supersonic turbulent boundary layer. (N.A.S.A. Ames Research Center.)

wall pressure structure strongly implies the relative insignificance of the noise it generates, at least at supersonic speeds. On the other hand, the same analysis applied to the sound produced by fluctuating wall shear stresses predicts the intensity to vary with M^4, the experimentally observed result. It requires a more detailed study of the boundary-layer radiation to establish its real source, for M^4 is also in encouraging agreement with the present theory which predicts M^3 without accounting for effects of varying scale in the direction of emission. As Mach number increases the Mach angle increases and the scale changes considerably—becoming smaller with increasing angle, so augmenting the radiation strength above the theoretical M^3 law.

Studies of acoustic fields generated by high-velocity turbulent jets reveal the complicated nature of the jet noise problem at high supersonic speeds. It would indeed be remarkable if the rapidly varying turbulent structure of a flow where the velocity changes so quickly in the presence of strong shock waves did not produce a complex noise field. Even in this situation some general aspects of the problem are seen to be in accordance with trends predicted by theory. The one large unknown in this respect is the location of the main noise-producing region and its associated convection velocity, if indeed any one convection velocity exists. The literature contains a number of reports where the most intense source of rocket noise is located a considerable distance downstream of the nozzle in a region of relatively slowly moving flow. The high-speed region near the nozzle evidently radiates with considerably lower efficiency. But these source locations have been estimated either by observations based on the directional maxima of the radiation field, a notoriously inaccurate process in complicated multipole systems, or by studies of the near-field pressure along the mixing flow boundary. This latter method is also likely to give a very misleading acoustic-source location in rocket exhausts where wide ranges of convection velocities are encountered. The reason is intimately connected with the order of the acoustic source and the fundamental change brought about by the simple-source-like emission of quadrupoles convected at sonic speed. Quadrupoles generally induce a powerful and highly localized 'near' field not present in simple-source emission. Measurements of pressure near a rocket exit would then be subject to this intense quadrupole 'near' field in the low-speed regions but only to the radiated field close to the supersonic flow where sonically-convected quadrupoles had degenerated into a simple-source system. The variation of pressure observed near the mixing flow of supersonic jets is very likely caused in part by this effect and it may be quite misleading to regard the 'near' field strength as an indication of the proximity of powerful acoustic sources.

While source locations remain in doubt the problem of understanding rocket noise fields is not so much one of alining a high-speed theory with the experimental results as one of establishing whether or not the rocket noise field is in the main generated by the high- or low-speed regions of the mixing flow. There is no doubt that for rockets acoustic power increases with velocity cubed, frequency spectra cease to be Strouhal number dependent and that the direction of peak emission moves towards the normal to the jet axis with increasing velocity (Cole, von Gierke, Kyrazis, Eldred & Humphrey 1957; Mayes, Lanford & Hubbard 1959; Lassiter & Heitkotter 1954), all effects entirely consistent with a supersonically-convected turbulent source region. But generally the degree of these effects are difficult to aline with any simple model of a supersonically-convected source structure. If rocket noise were mostly of the Mach wave type it would be extremely directional with a spectrum independent of jet velocity but at generally higher frequencies than those present in subsonic jets. Lassiter & Heitkotter (1954) found relatively low frequencies in their rockets and although the noise field was very directional it was not as directional as the example of § 5, if indeed this is at all realistic, would predict. Eldred (1956), Cole *et al.* (1957), Mayes *et al.* (1959) all show that for rocket noise a Strouhal number based on jet velocity is no longer useful in normalizing acoustic spectra and they use a Strouhal number based on the relatively slowly-changing velocity of sound at the nozzle exit. A possible explanation for this apparently poor correlation is that acoustic efficiency has reached such a high value at high speeds that turbulence levels are damped by the heavy energy demand of the radiation field. This however is a most unlikely explanation since experimental observation shows that jet and rocket motors rarely exceed an acoustic efficiency greater than $\frac{1}{2}$ % (Mayes *et al.* 1959). This is brought home very vividly when one considers that at subsonic speeds approximately 80 % of the jet kinetic energy is dissipated in the region thought to contain the noise-producing eddies (see Corrsin 1946; Lilley 1958) and a similar situation very likely exists at higher speeds. Most of this energy passes through the turbulence phase and would be available for sound generation were the mechanism of energy conversion of high efficiency. Even so, since only a very small fraction of the turbulence spectrum, $H_{ijkl}(\boldsymbol{\eta}, \mathbf{k}, \omega)$, is capable of exciting acoustic waves, high-efficiency radiation could reduce significantly the energy in that region yet have a negligible effect on the overall turbulence level. This could account for some features of the sound field which imply acoustic damping is active, such as departures from the basic U^8 law at $\theta = 90°$ and the reduced spectrum range of rocket noise, yet be consistent with the observation that the radiated energy is a negligible fraction of the energy dissipated in the sound-producing region.

But a further argument that acoustic damping is not an active mechanism can be based on the few results which show the strongest resemblance to the predicted high-speed radiation field. Cole *et al.* (1957) report an experiment where a flame inhibitor was added to the rocket exhaust which prevented burning of excess fuel in the supersonic shear region. The noise field of this test is exceedingly directional and the spectrum contains a considerably augmented high-frequency power. In fact this noise field provides the best qualitative agreement with high-speed theory available so far, yet addition of the flame inhibitor increased the noise level by a factor of four! The reason for the relative lack of high-speed features in most rocket results must be, not that those features only occur in a radiation field making an impossibly high-energy demand on the turbulence, but that the burning of excess fuel in the exhaust modifies the turbulence structure to such a degree that the sources become most inefficient. This test also serves as an indication of the predominance of the fluctuating Reynolds stress in the source strength density. Burning of excess fuel in the mixing-region would cause large temperature inhomogeneities, but these effects are quite negligible since burning seems to reduce noise rather than increase it. The other example of a high-speed jet flow devoid of combustion in the wake is the high-power turbojet engine. Evidently† its noise field exhibits a frequency spectrum quite independent of exhaust velocity at the condition where intensity assumes a dependence on velocity cubed. This is again fully in accordance with a noise field produced by a supersonic source region.

† Private communication with Mr G. M. Coles of Rolls Royce Ltd.

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But before we treat this evidence as confirmation of the high-speed theory we must first ask whether or not these effects can be accounted for if the subsonic regions were mainly responsible for the noise? One might postulate that then the noise field would be closely similar to that of a jet with sonic exit velocity but with a scale sufficiently large to accommodate the increased mass flow. The area of the sonic section would have to vary with the square of jet velocity and so cause the noise level, directly proportional to area, to increase with velocity squared at high speed. Frequencies radiated by such a system would vary inversely with scale and hence inversely with jet velocity while the directionality would be that of a subsonic jet. Experimental evidence does not support a model of this type although only a

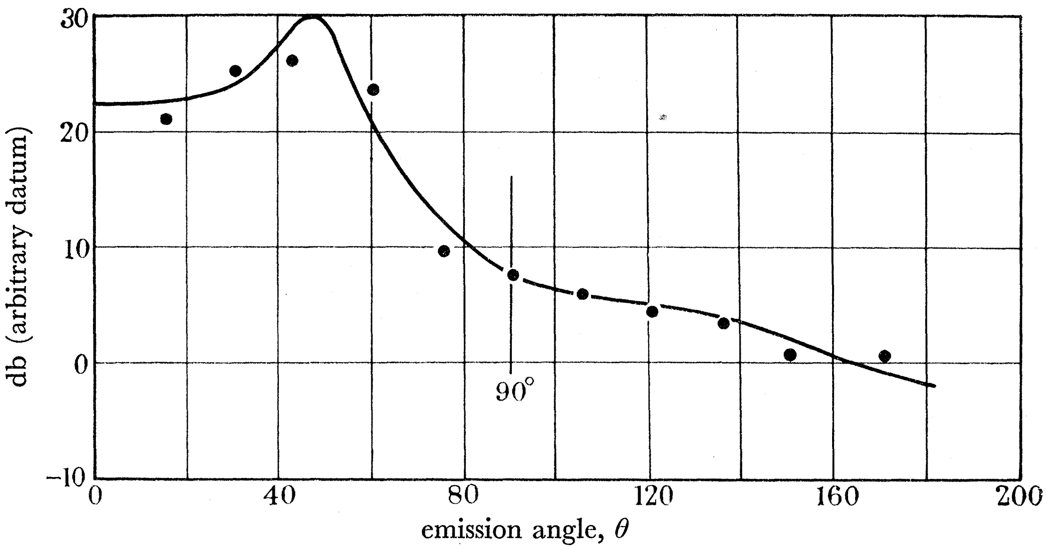

FIGURE 14. Comparison of measured directionality with a theoretical model.
●, Lassiter & Heitkotter (1954); solid fuel rocket exit velocity 5340 ft./s.

relatively small proportion of noise arising from a supersonic region is necessary to account for the measured directionality. On the basis that acoustic sources in the supersonic region are convected at one velocity and those in the subsonic region with half the speed of sound one could combine two directional characteristics corresponding to the two eddy speeds to simulate the choked jet case. This is done in figure 14 and compared with the measurements obtained by Lassiter & Heitkotter (1954) with a solid fuel rocket motor. The relative contributions from the supersonic and subsonic sections has to be estimated, but if the ratio is 1:32 and the supersonic convection takes place at $M = 1{\cdot}5$ the comparison with the measurements is remarkably good. Here the example of § 5 has been employed and the ratio of lateral to longitudinal quadrupoles is that suggested for low-speed jets by Lilley (1958). It is highly probable that this agreement is purely fortuitous since it is very difficult to explain why only 3 % of the noise is generated by the supersonic region yet the intensity is known to be increasing with velocity cubed while the frequency spectrum is changing only very slowly. Only sound emitted by the high-speed flow varies in this manner, and for the overall noise to exhibit these properties implies very strongly that it is mostly generated in the supersonic region.

One final rocket experiment which is interesting in that it implies the relative insignificance of strong shock waves in the mixing flow is reported by Mull & Erickson (1957).

Ffowcs Williams & Ricketson (1963) in discussing this work remark that 'using a constant chamber pressure and nozzle area they find the total acoustic power from an isentropic nozzle, a 15° half angle conical nozzle and a convergent nozzle was 99 watts, 78 watts and 26 watts respectively. It would be expected that the isentropic nozzle having only weak shocks in the flow would have the longest supersonic core while the convergent nozzle having large normal shocks would have the shortest.' Could it be that the noise is mostly generated in the supersonic core and that the only significant part played by strong shock waves is to reduce the volume containing powerful acoustic sources? This and many other questions must for the present remain unanswered. With more detailed experiments our knowledge of the mixing region and its acoustic sources will increase, making quantitative checks of the theory possible, but it would be quite misleading at this stage to be emphatic on apparent agreement of experiment and theory. That some features observed experimentally are in clear agreement with theory provides some encouragement that the techniques developed by Lighthill (1952, 1954) which have been so successful at low speeds can be extended to form a theoretical basis for rocket-noise studies.

The work described above has been carried out as part of a research programme at the Aerodynamics Division of the National Physical Laboratory, and this paper is published by permission of the Director of the Laboratory.

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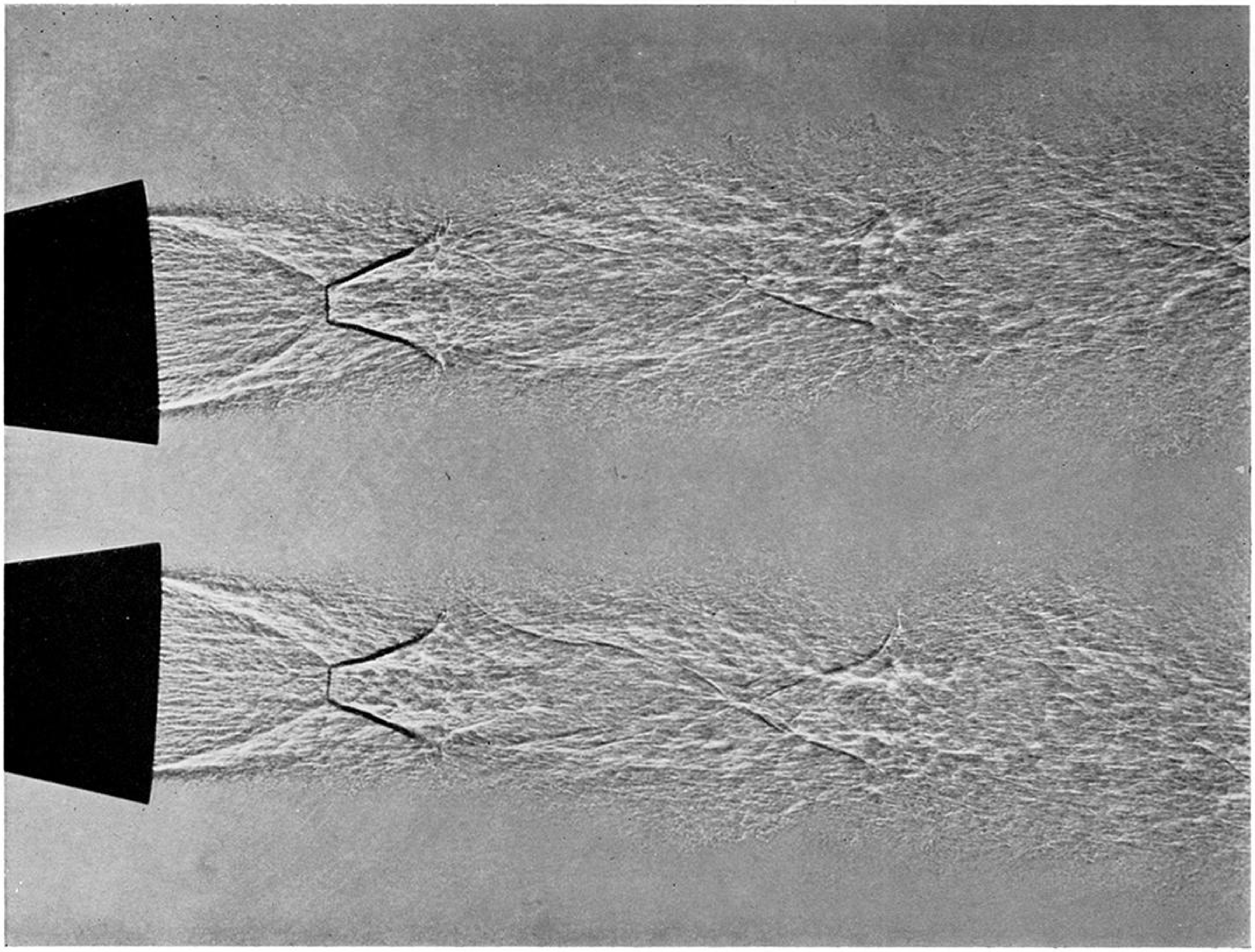

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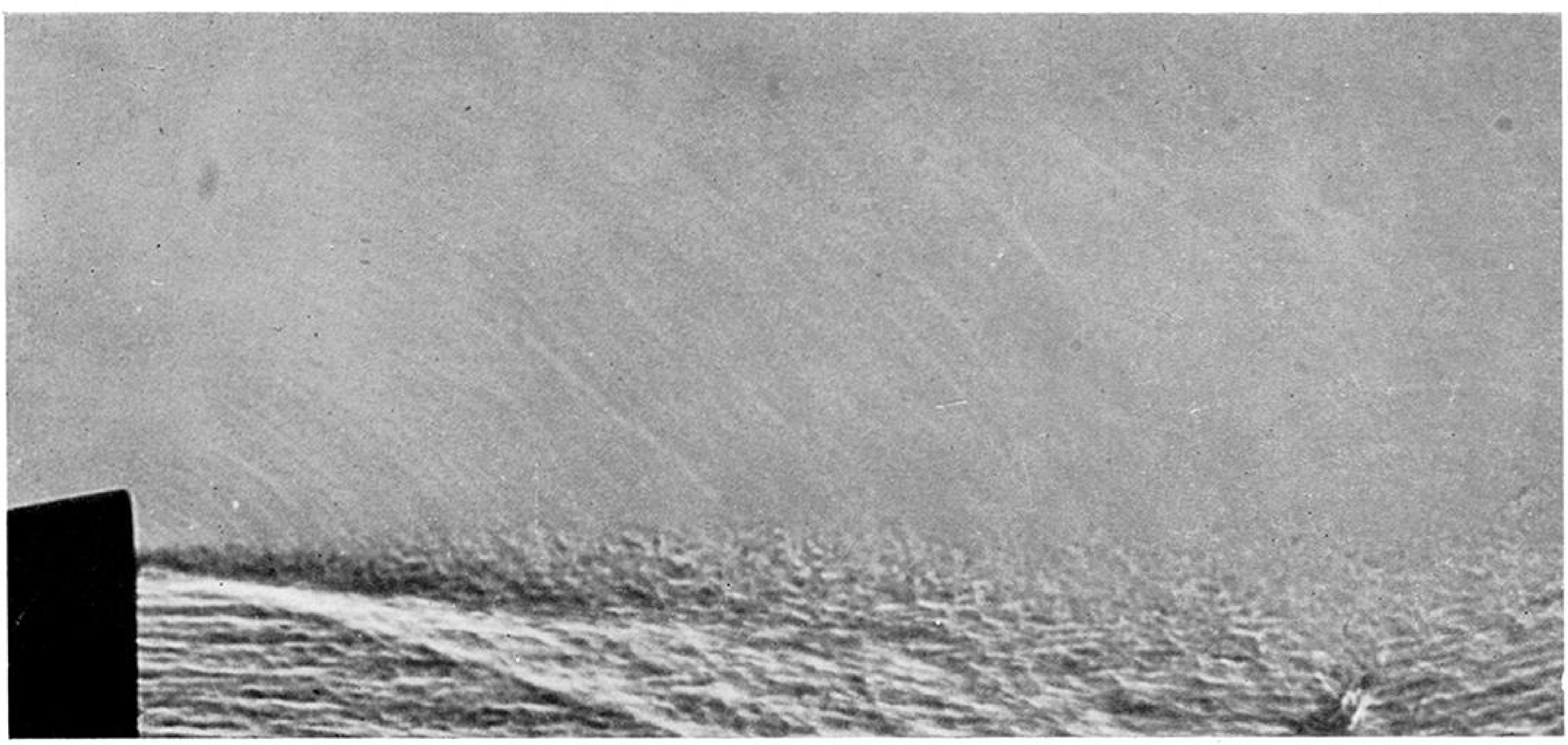

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FIGURE 12. Shadowgraph pictures illustrating Mach wave radiation generated by the exhaust of high-speed jets. (Crown copyright is reserved and this photograph is published by permission of Her Majesty's Stationery Office.)

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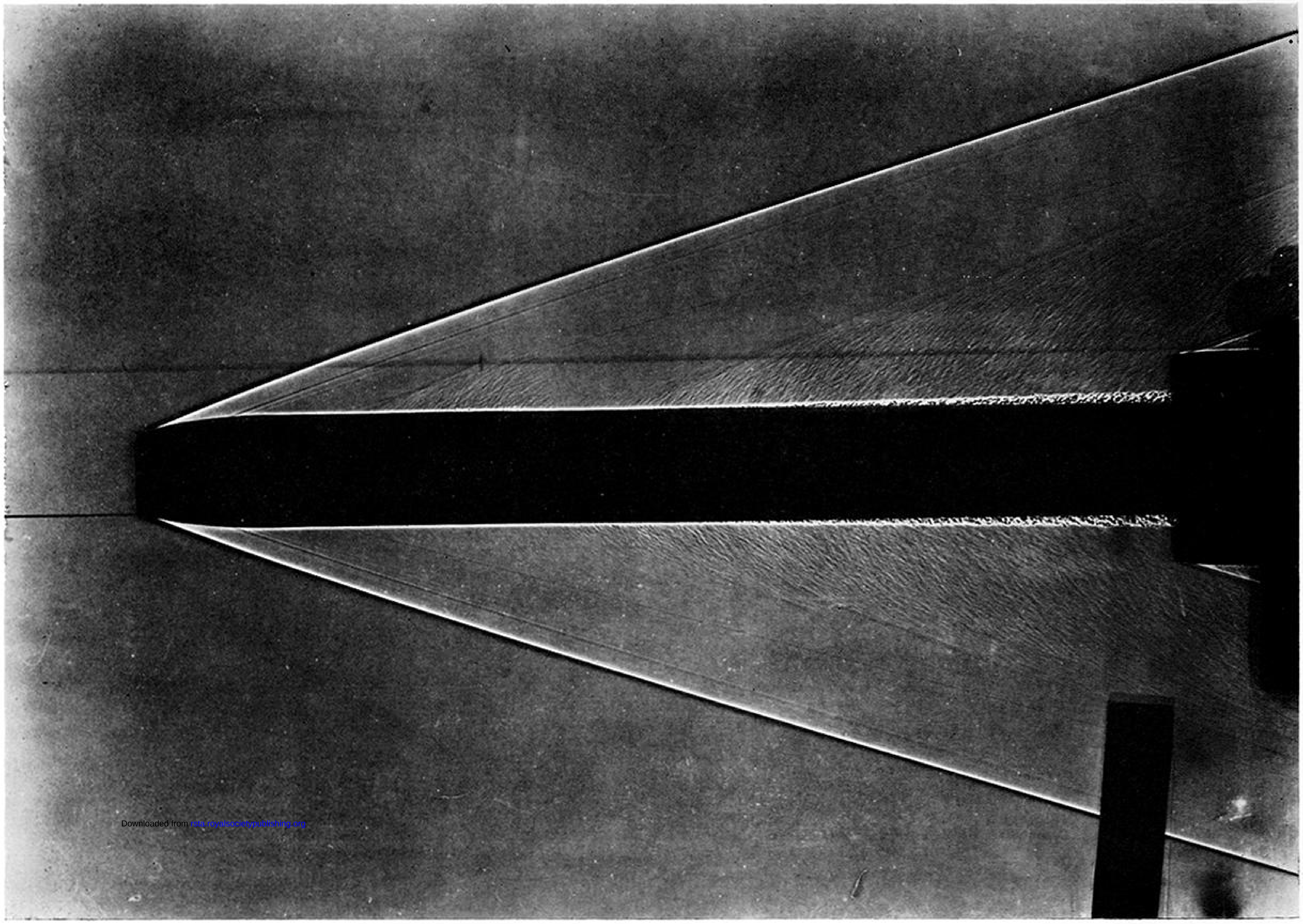

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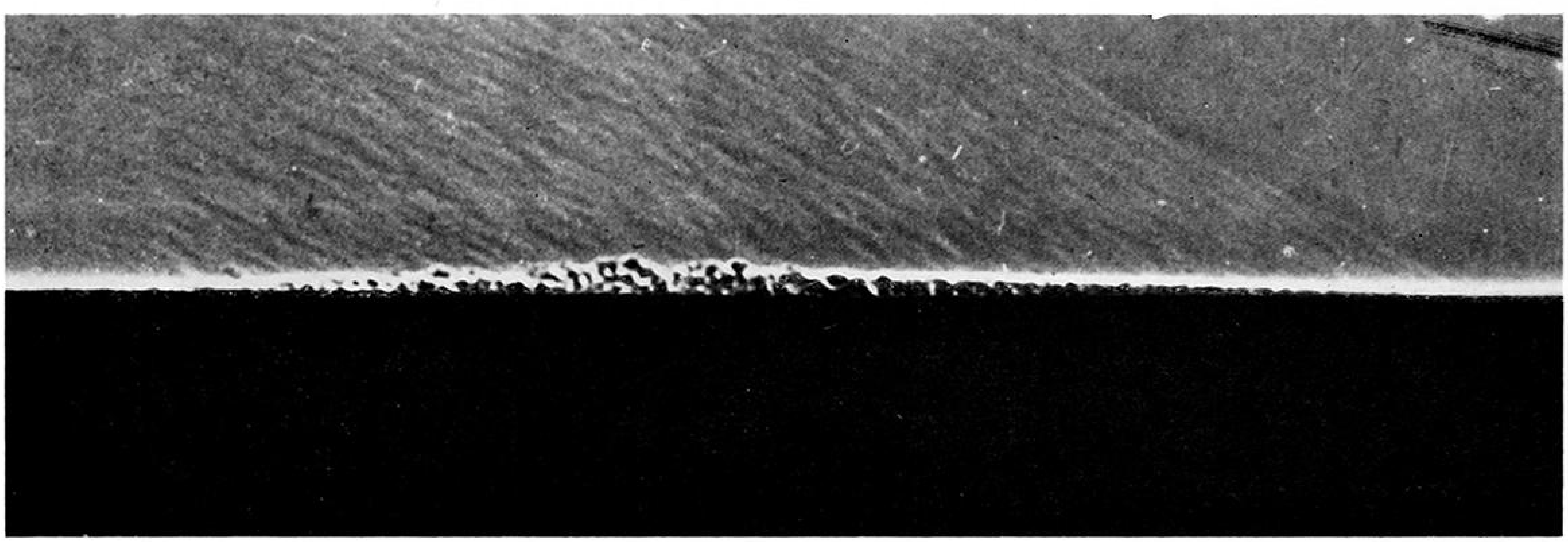

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FIGURE 13. Shadowgraph pictures illustrating Mach wave radiation generated by a supersonic turbulent boundary layer. (N.A.S.A. Ames Research Center.)

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